

THREE TRACTS;
VIZ.
AN INTRODUCTION
TO THE
DOCTRINE OF FLUXIONS.
By JOHN ROWE.

MATHEMATICS.
By the late rev.^d Mr. WM. WEST.
Revised, altered & enlarged, by J. ROWE.

LETTERS,
Relative to SOCIETIES
for the BENEFIT of
WIDOWS and of AGE.
By J. ROWE.

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AN
INTRODUCTION
TO THE
DOCTRINE
OF
FLUXIONS.

With THIRTEEN COPPER-PLATES.

By JOHN ROWE. <

The THIRD EDITION:
With *Additions and Alterations.*

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☞ Extract of a Letter to the Author, from the learned and well-known—the Reverend J. STUBBS, M. A. Fellow of Queen's College, Oxford.—Dated, Feb. 16. 1771.

“ I received your valuable present, and was much surprised to
“ find it so prodigiously improved. Indeed, it so much resembles a
“ New Work, when compared with the First Edition, that I
“ almost wish you had made no mention of its being the Third;
“ but left the two former to be forgotten.”

T O
WILLIAM DAVY Esq;

One of His MAJESTY's
SERJEANTS at LAW,

T H I S
T R A C T

IS INSCRIBED,

By his most Obedient

Humble Servant

The AUTHOR.



T H E

P R E F A C E.

OF all the Mathematical Sciences, *The Doctrine of Fluxions* is the most extensive and sublime. By this, many Difficulties, unsurmountable by any other known Method, are solved with uncommon Expedition, Elegance, and Ease.

It is a General Way—for determining the *maxima* and *minima* of Quantities; drawing Tangents to Curves, finding their Points of Inflection and Radii of Curvature:—for obtaining the Lengths of curve Lines, the Areas of curvilinear Spaces, the Surfaces and Solidities of concave and convex Bodies: &c. —In a Word, It extends to the investigating the most abstruse and difficult Problems in
the

the various Branches of mathematical and philosophical Science.

The *Method of Fluxions* was first invented in the Years 1665 and 1666, by that Prince of Mathematicians and Philosophers the late Sir *Isaac Newton*, then Mr. *Newton* about 23 Years old *. It was soon after communicated to some of his Friends; but, he gave no public Specimens thereof until the appearance of his immortal *Philosophiæ Naturalis Principia Mathematica*, printed in the Year 1687: The celebrated German therefore, Mr. *Godfrey William Leibnitz* † to whom it was hinted in 1676, applied it in the *Acta Eruditorum*, printed at *Leipsic* in 1684, to a few Problems de *Maximis et Minimis* and Tangents to Curves, and claimed the Invention himself ‡.

——— Their *Notations* indeed are different; and, Quantities being by both, in Effect, considered as produced by continual Increase after the same manner as Space is described
by

* He was Born December 25. 1642, Knighted in 1705, and Died March 20. 1726.

† See *Raphson's History of Fluxions*, printed in the Year 1715; or, the *Commercium Epistolicum*, published by Order of the Royal Society in 1722; wherein, Sir *Isaac* is fully proved the Original Inventor of this noble and most delightful Method.

‡ He was born at *Leipsic* June 23. 1646, and died Nov. 14. 1716.

by a Body in Motion; instead of the *Velocity* with which a Quantity varies or flows at any Point or Term of the Time in which it is supposed to be generated, called by Sir *Isaac* a *Fluxion*, *Leibnitz* takes the *Increment*, or little Part generated in an indefinitely small Portion of Time, and calls it a *Differential**.

Several excellent Treatises have been published on the Subject; but, as they appear not calculated to introduce the young and unassisted *Beginner* into this abstruse and difficult Science, in order to his understanding them a plain and easy *Introduction* seems to be necessary; and for that End the following Sheets are chiefly designed.

This Tract is divided into *three* Parts: the *first* treats of the *Direct* Method of Fluxions; in which, from the generated Quantity or *Fluent* being given, we find the *Fluxion*: and the *second* of the *Inverse* Method; wherein, from the *Fluxion* being known,
we

* *Leibnitz* denotes the *Differential* of any variable quantity x by dx : and Sir *Isaac*, generally, for it's *Fluxion* writes $\dot{x}$; but in his *Principia* flowing quantities are expressed by the capital letters, A, B, &c. and their *Fluxions* or *Increments* by the corresponding small letters a , b , &c.

we find the *Fluent* *: the *third* contains *miscellaneous Questions* with their *incremental* and *fluxional Solutions*; which could not, with propriety, be inserted in the former Parts; and to some of which there was occasion to refer.

In this *Third Edition* † are many *Additions* and *Alterations*. The *Constructions* are, in general, *New* ‡. In a Word, It contains, perhaps, a variety of Things not to be found in any other Tract on the Subject.

In order to a thorough understanding of this *Introduction*, it is requisite that the Learner be well acquainted with Arithmetic, Algebra, Geometry, Plane-Trigonometry, Conic-Sections, and the Nature of Logarithms. But, as geometrical and algebraical Treatises, in general, give not the Descriptions, and from thence the deduction

* The *Direct Method* of Fluxions, as delivered by *Leibnitz*, is, by *Foreigners*, called *Calculus Differentialis*; and the *Inverse Method*, *Calculus Integralis*.

† The *First Edition* was printed in the Year 1751; and the *Second*, with *Alterations* and *Additions*, in 1757.—Cuts.

‡ Those in *art.* 35, 39, 78, and 86, may be seen in other Books.

The PREFACE.

ix

duction of the Properties, of some Curves to be found in the following Sheets; nor the Methods of reducing Quantities into Infinite Series, and of Noting their Powers and Roots, necessary to be used in *fluxional* Tracts; therefore, these Deficiencies are herein supplied, though they do not immediately relate to the Business in Hand.

JOHN ROWE.

January 8. 1767.

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to be found in the following
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John Howe.

January 31 1767.

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T H E
C O N T E N T S.

P A R T I.

CHAP. I.	<i>Of the Principles of Fluxions, and of the New Notation in Algebra</i>	Page 1
CHAP. II.	<i>Of finding the Fluxion of a given Fluent</i>	13
CHAP. III.	<i>Of drawing Tangents to Curves</i>	28
CHAP. IV.	<i>Of finding the Maxima and Mini- ma, or Greatest and Least of variable Quantities</i>	56
CHAP. V.	<i>Of finding the Points of Inflection in Curves</i>	87
CHAP. VI.	<i>Of finding the Radius of Curva- ture</i>	101
CHAP. VII.	<i>Of finding the Nature of the E- volve of a given Involute Curve</i>	124

P A R T

P A R T II.

CHAP. I. <i>Of Infinite Series</i>	Page 133
CHAP. II. <i>Of finding the Fluent of a given Fluxion</i>	148
CHAP. III. <i>Of finding the Length of a Curve Line</i>	158
CHAP. IV. <i>Of finding the Areas of Curvilinear Spaces</i>	168
CHAP. V. <i>Of finding the Convex Superficies of Solids</i>	175
CHAP. VI. <i>Of finding the Contents of Solids</i>	180

P A R T III.

MISCELLANEOUS QUESTIONS, <i>with their Incremental and Fluxional SOLUTIONS</i>	189
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A N
INTRODUCTION
TO THE
DOCTRINE
O F
FLUXIONS.

PART I.

CHAP. I.

*Of the Principles of Fluxions, and of the
New Notation in Algebra.*

1. **I**N this Doctrine, Quantities are supposed to be generated by continual increase, after the manner of a Space which a body in motion describes.

Thus, a Line is supposed to be generated by a Point in motion, a Superficies by a Line, and a Solid by a Superficies.

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2. THE

2. THE Velocity with which a Quantity flows, or is generated, at any particular point or term of it, is called the *Fluxion* of that Quantity at that point or term.

Fig. Thus, if we suppose the indefinite right
 1. line AZ to move with a parallel mo-
 2. tion along the axis AX , or, so as always
 to be parallel to it's first situation; and, at
 the same time, a point to move from A
 along the said line AZ so as to generate
 or always to be in the curve AY : Then,
 the Velocity with which the end or point A
 of the line AZ arrives at any point C , or,
 which is the same, the Velocity with which
 the axis flows or is generated at any parti-
 cular point C , is called the Fluxion of the
 axis at that point; and the Velocity with
 which the point moves along the line AZ
 at any point B , that is, the Velocity with
 which the ordinate flows or increases at any
 point B , is called the Fluxion of the ordi-
 nate at that point; also, the Velocity with
 which the point generates or moves along
 the curve at any point B , is called the
 Fluxion of the curve at that point; like-
 wise, the Velocity, or degree of quickness, with
 which the curvilinear space ACB flows or
 is generated by the line AZ at any term
 CB ,

CB, is called the Fluxion of the said curvilinear space at that term.

3. Now, if the Velocity with which any quantity flows, or is generated, be at every point or term the same; that is, if it be neither accelerated nor retarded; the Fluxion of it will likewise be at every point or term the same. But if this Velocity be continually increased or diminished, then there will be a certain degree of Velocity, or Fluxion, peculiar to every point or term of the thing described: And the Velocity wherewith the said velocity, at any point or term, is either accelerated or retarded, is called the *Fluxion of the Fluxion*, or the *Second Fluxion*. And, again, if this acceleration or retardation be not uniform, but is continually varying; or the velocity with which the quantity flows does not uniformly increase or decrease; then, the Velocity, or degree of swiftness, with which this acceleration or retardation either increases or decreases, is called the *Third Fluxion*. And so on.

4. THE indefinitely small Increase of a Quantity, generated in an indefinitely small particle of time, is called the *Increment* of that Quantity.

4 *An INTRODUCTION to the*

Fig. Thus, if we suppose bc indefinitely
 3. near and parallel to the ordinate BC , and
 4. Bn parallel to the absciss AC ; then, Cc , or
 it's equal Bn , is called the Increment of the
 absciss AC ; nb the Increment of the ordi-
 nate CB ; Bb the Increment of the curve
 AB ; and $CBbc$ the Increment of the
 curvilinear space ACB .

5. Now, if ed be supposed indefinitely
 near and parallel to bc , and br equal and
 parallel to Bn or cd ; then, the Difference
 between nb and re is called the *Increment of*
the Increment, or the *Second Increment*; that
 is, the *Increment of nb* , or *Second Increment*
 of CB . And, again, if sf be supposed in-
 definitely near and parallel to ed , and et
 equal and parallel to br or df ; then, the
 Difference between the Second Increment
 and that of re and ts , is called the *Third*
Increment of CB . And so on.

6. *Note.* When a Quantity, instead of
 increasing, is continually diminished; then,
 the indefinitely small Particles by which it is
 lessened, are not, properly, called *Increments*,
 but *Decrements*. And both *Increments* and
Decrements are sometimes called *Moments*.

7. Now,

7. Now, if we suppose the absciss, AC , to flow on with an uniform motion, or equal parts of it to be generated or described in equal times; it's Increment will accurately express or be exactly as it's Fluxion; since velocity is always expressed by, or is as the space uniformly described in a given time: And therefore, if the Curve Bb did exactly coincide with the Tangent or right line TBG , it is evident, that then, the Increments Bb and bn would likewise be described with uniform motions, and the same degrees of velocity with which the curve and ordinate respectively flow at the point B ; that is, the Increments Bb and bn would then be accurately as the Fluxions of the curve and ordinate at the point B . But, since not two points of the Curve are coincident with the Tangent, and consequently the velocities with which the Increments are generated are continually varying in every point; therefore, the Increments and Fluxions are not in an exact proportion to each other; or, the Increments do not accurately measure the Velocities or Fluxions with which they begin to be generated. However, as the point b is continually nearer to a coincidence with the Tangent GB the nearer it ap-

6 *An* INTRODUCTION to the

approaches the point of contact B; so therefore, if we conceive the ordinate cb to move back until it coincides with CB ; then, the very first moment before it's coincidence, the curve Bb and right line BG will be infinitely or rather indefinitely near to a coincidence with each other; and consequently, in that case, the Increments Bb and bn will come indefinitely near to measure the Fluxions of the curve and ordinate, or the Velocities with which they flow at the point B: Or, because the particles of time in which any Increments are generated are supposed to be indefinitely small, and consequently, the acceleration or retardation of the Velocities with which they are generated must be so too; therefore, they are indefinitely near in proportion to the Fluxions of the quantities of which they are Increments: But, when *Ratios*, from that of Equality, are but indefinitely little, or less than can be assigned, they may be considered as Equal *. Hence therefore, the Increments may be taken as proportional to, or for the Fluxions in all operations; and, on the contrary, the Fluxions for the Increments.

8. THOSE

* This was allowed by the ancient Geometricians, *Euclid*, *Archimedes*, &c.

8. THOSE quantities which are supposed to flow, or to be generated by continual increase, as the absciss and ordinate of a curve, are called *Fluents*, and *variable* or *flowing* quantities: And those which neither increase nor decrease, or admit of no variation, as the parameter of a conic-section and the diameter of a circle, are called *fixed*, *given*, and *invariable* quantities.

9. THE Beginning of the Alphabet, viz. *a, b, c, &c.* is used to express Invariable quantities; and the End of it, viz. *x, y, z, &c.* Variable or Flowing quantities.

10. THE Fluxion of any variable quantity x , is denoted by $\dot{x}$; it's Second Fluxion, or the Fluxion of $\dot{x}$, by $\ddot{x}$; it's Third Fluxion, or the Fluxion of $\ddot{x}$, by $\dddot{x}$; and so on. Also, the Moment Increment or Decrement of x , is denoted by x' ; it's Second Moment Increment or Decrement, or the Moment Increment or Decrement of x' , by x'' ; and so on.

11. THE Fluxions and Moments of Invariable quantities, viz. of *a, b, c, &c.* are evidently $= 0$.

12. THOSE Fluents which are generated in the same time, or in equal times, or which begin together and end together, are called

con-

8 *An* INTRODUCTION to the

contemporary Fluents; and the Fluxions of these contemporary Fluents are called *contemporary* Fluxions. Now, it is evident, if two or more of these contemporary Fluents are always equal, or in any invariable ratio to each other, that their contemporary Fluxions will likewise be equal, or in the same proportion: And that, on the contrary, if two or more of these contemporary Fluxions are always equal, or in any invariable ratio to each other, their contemporary Fluents will likewise be equal, or in the same proportion.

WE come now to find the *Fluxions* of *Fluents*, or the Velocities with which Flowing quantities increase or decrease at any points or terms assigned: the Business of which is called the *Direct* Method of Fluxions. But, before we proceed, it may, perhaps, be Necessary to treat of what is called

THE NEW METHOD OF NOTATION IN
ALGEBRA *.

13. IN Surds, the Index shewing the Height of the Power to which any given quantity

* This was invented by the late celebrated Dr. Wallis, and first published in his *Arithmetica Infinitorum* in the Year 1656.— He was born Nov. 23. 1616, and died Oct. 1703.

DOCTRINE of FLUXIONS. 9

quantity is to be raised, is here placed as the Numerator of a Fraction, whose Denominator is the radical sign or Index shewing the Root to be extracted *.

Thus, $\sqrt[3]{x^2}$ is expressed by $x^{\frac{2}{3}}$, and $\sqrt{a+x}$ by $(a+x)^{\frac{1}{2}}$.

Also, $\frac{1}{\sqrt[3]{x^2}}$ is expressed by $\frac{1}{x^{\frac{2}{3}}}$ or $x^{-\frac{2}{3}}$, and $\frac{1}{\sqrt{a+x}}$ by $\frac{1}{(a+x)^{\frac{1}{2}}}$ or $(a+x)^{-\frac{1}{2}}$. §

Now, in any geometrical progression, whose first term is *unity*, or 1; if we take an arithmetical progression, whose first term is 0, and whose second term, or common difference, is the Index of the quantity in the second term of the geometrical progression; then, the number in any term of the arithmetical progression will be the Index of the quantity in the corresponding term of the

* This Fraction is called the *Index* of the Power, and the given quantity itself the *Root* of the Power.

§ The propriety of using *negative* Indices is evident; for, to divide any Power of x by x , is only to lessen the Index of the Power by 1; or, to subtract the Index of the Denominator from that of the Numerator. Thus, $\frac{x^3}{x^1}$ is $= x^2$; $\frac{x^2}{x^1}$ is $= x^1$ or x ; $\frac{x^1}{x^1}$ is $= x^0 = 1$; $\frac{x^0}{x^1}$ is $= x^{-1}$, that is, $\frac{1}{x}$ is $= x^{-1}$; &c.

the geometrical progression: And therefore, the arithmetical mean between the numbers in any two terms of the arithmetical progression will be the Index of the geometrical mean between the quantities in the two corresponding terms of the geometrical progression.

Thus, in the geometrical progression $1.x^{\frac{1}{2}}.x^1.x^{\frac{3}{2}}.x^2.x^{\frac{5}{2}}.x^3.$ &c. where the common multiplier is $x^{\frac{1}{2}}$; if we take the arithmetical progression $0.\frac{1}{2}.1.\frac{3}{2}.2.\frac{5}{2}.3.$ &c. wherein the common difference, or quantity added, is $\frac{1}{2}$; the number in either term of the arithmetical progression is the Index of the quantity in the corresponding term of the geometrical progression: And, the arithmetical mean between the numbers in any two terms of the arithmetical progression is the Index of the geometrical mean between the quantities in the two corresponding terms of the geometrical progression. For instance, $\frac{3}{2}$ the 4th term of the arithmetical progression, is the Index of $x^{\frac{3}{2}}$, the 4th term of the geometrical progression: And, the arithmetical mean between $\frac{1}{2}$ and $\frac{5}{2}$, the 2d and 6th terms

terms of the arithmetical progression, which is $\frac{1}{2}$, is the Index of the geometrical mean between $x^{\frac{1}{2}}$ and $x^{\frac{3}{2}}$, the same two terms of the geometrical progression, which is $x^{\frac{1}{2}}$.

Or, in the descending geometrical progression, or series, $1 \cdot \frac{1}{x^{\frac{1}{2}}} \cdot \frac{1}{x} \cdot \frac{1}{x^{\frac{3}{2}}} \cdot \frac{1}{x^2} \cdot \frac{1}{x^{\frac{5}{2}}} \cdot \frac{1}{x^3} \cdot \&c.$

that is, $1 \cdot x^{-\frac{1}{2}} \cdot x^{-1} \cdot x^{-\frac{3}{2}} \cdot x^{-2} \cdot x^{-\frac{5}{2}} \cdot x^{-3} \cdot$

$\&c.$ where the common divisor is $x^{\frac{1}{2}}$; if we take the arithmetical progression, or series,

$0 \cdot -\frac{1}{2} \cdot -1 \cdot -\frac{3}{2} \cdot -2 \cdot -\frac{5}{2} \cdot -3 \cdot \&c.$ wherein the

common difference, or quantity subtracted, is $\frac{1}{2}$; the number in any term of the arithmetical series, is the Index of the quantity in the corresponding term of the geometrical series: thus, $-\frac{3}{2}$, the 4th term of the arithmetical series, is the Index of $x^{-\frac{3}{2}}$, the same term of the geometrical series. And, the arithmetical mean between any two terms of the arithmetical series, is the Index of the geometrical mean between the same two terms of the geometrical series: as, for instance, the arithmetical mean between $-\frac{1}{2}$ and $-\frac{5}{2}$, the 2d and 6th terms of the arithmetical series, which is $-\frac{3}{2}$, is the Index

of the geometrical mean between $x^{-\frac{1}{2}}$ and $x^{-\frac{5}{2}}$, the 2d and 6th terms of the geometrical series, which is $x^{-\frac{3}{2}}$.

HENCE we may observe, that, in the Indices of Powers, Addition has the effect of Multiplication on the respective Roots, and Multiplication of Involution; and, *e contra*, Subtraction of Division, and Division of Evolution: or that, in a word, Indices of Powers are entirely Logarithmical with regard to their Roots.

So that, $x^2 \times x^{\frac{3}{2}}$ is $= x^{2+\frac{3}{2}} = x^{\frac{7}{2}}$, $\overline{x^3}^{\frac{2}{3}}$ is $= x^{3 \times \frac{2}{3}} = x^2$, $\frac{x^5}{x^3}$ is $= x^{5-3} = x^2$, $\overline{x^4}^{\frac{1}{2}}$ is $= x^{\frac{4}{2}} = x^2$; and $x^2 \times x^{-\frac{3}{2}}$ is $= x^{2-\frac{3}{2}} = x^{\frac{1}{2}}$, $\overline{x^2}^{-3}$ is $= x^{2 \times -3} = x^{-6}$, $\overline{x^4}^{-\frac{1}{2}}$ is $= x^{4 \times -\frac{1}{2}} = x^{-2}$: or, Universally, $x^m \times x^n$ is $= x^{m+n}$, $\overline{x^m}^n$ is $= x^{mn}$, $\frac{x^m}{x^n}$ is $= x^{m-n}$, and $\overline{x^m}^{\frac{1}{n}}$ is $= x^{\frac{m}{n}}$; where note, m and n represent any affirmative or negative whole-numbers or fractions whatever *.

If

* These Indefinite Indices were introduced by the great Inventor of Fluxions.

If what has been said be duly considered, no difficulty in this *New Notation* will occur; the knowledge of which is absolutely necessary, in order to the well understanding the following pages.

C H A P. II.

Of finding the Fluxion of a given Fluent.

R U L E I.

14. To find the Fluxion of a Simple Fluent, or, of that wherein there is but One variable letter or flowing quantity.

MARK the variable letter or flowing quantity with a dot over it: and you will have the Fluxion required.

Thus, the Fluxion of ax is $= a\dot{x}$.

For, if we suppose the variable rectangle AB to be generated by the given line CB setting out from the situation AD and moving along with a parallel motion between the parallel and indefinite sides AC and DB ; it is evident, the Velocity with which the rectangle flows is equal to the generating line CB drawn into the Velocity with which the point C generates or moves along the line AC : Fig. 5.

14 *An* INTRODUCTION to the

AC: that is, the Fluxion of the rectangle AB is equal to the invariable line CB drawn into the Fluxion of the flowing or variable line AC. Therefore, if AD or CB = a , and AC or DB = x ; then, the Fluxion of the rectangle ax will be $= a \times \dot{x} = a \dot{x}$.

R U L E II.

15. To find the Fluxion of the Product of two or more flowing quantities drawn into each other.

MULTIPLY the Fluxion of each quantity separately, by the other or the product of the rest of the quantities: and the Sum of these products will be the Fluxion required *.

Thus, the Fluxion of xy is $= \dot{x}y + x\dot{y}$; the Fluxion of xyz is $= \dot{x}yz + x\dot{y}z + xy\dot{z}$; and the Fluxion of $vxyz$ is $= \dot{v}xyz + v\dot{x}yz + vx\dot{y}z + vxy\dot{z}$.

Fig. 1°. That the Fluxion of xy is $= \dot{x}y + x\dot{y}$, may thus be proved. Suppose
6. a

* The Common Methods of proving the truth of this Rule, which are by the aid of Increments, were smartly attacked by the late acute Dr. Berkeley, Bishop of Cloyne, in a Pamphlet called *The Analyst*, printed in the Year 1734: But, his Objections, it is presumed, are by no means applicable to the demonstrations here given: On the contrary, it is not doubted, but the reasoning here advanced will be allowed to be scientific, fair and conclusive.

a line, coincident with the indefinite right line AF , to move with a parallel motion along the indefinite right line AE perpendicular to AF ; and at the same time, another, coincident with the line AE , to move with a parallel motion along the line AF ; and that the said two lines move with such different degrees of velocity as that their points of intersection be always in the curve AI . Through any point B in the curve, draw CH parallel to AF , and DG parallel to AE . Then, by the line moving along the line AE will the curvilinear space AEI and rectangle DE be generated; and by the line moving along the line AF the curvilinear space AFI and rectangle CF will be generated. Now, before the line moving along the line AE arrives at the situation CH , it is evident, that, the curvilinear space AEI will increase slower, or, flow with a less degree of velocity, than the rectangle DE , and afterwards faster, or with a greater degree of velocity; therefore, at the term CB , they will flow or increase with one and the same degree of velocity: So likewise, the curvilinear space AFI will flow or increase slower, or with a less degree of velocity, than the rectangle CF , before the generating line, moving

ing along the line AF , comes to the situation DG ; and, afterwards, faster, or with a greater degree of velocity; and therefore, at the term DB , they will increase or flow with an equal degree of velocity. That is, the Fluxion of the curvilinear space AEI is, at the term CB , equal to the Fluxion of the rectangle DE at the said term CB ; and, the Fluxion of the curvilinear space AFI is, at the term DB , equal to the Fluxion of the rectangle CF at the same term DB . But, (*art. 14.*) the Fluxion of the rectangle DE , at the term CB , is equal to CB drawn into the Fluxion of AC ; and the Fluxion of the rectangle CF , at the term DB , is equal to DB drawn into the Fluxion of AD . Hence therefore, the Fluxion of the flowing rectangle $ACBD$ (or of the sum of the two curvilinear spaces ACB and ADB) is equal to CB drawn into the Fluxion of AC , added to DB drawn into the Fluxion of AD : that is, if we put AC or $DB = x$, and AD or $CB = y$, the Fluxion of the rectangle xy will be $= \dot{x}y + x\dot{y}$.

2°. And, that the Fluxion of xyz is $= \dot{x}yz + x\dot{y}z + xy\dot{z}$, may thus be proved. Let the length, breadth, and depth, of a parallelopipedon, be represented by x , y , and z ,

re-

respectively. Now, this parallelopipedon will be equal to three pyramids, whose bases are xy , xz , and yz , and altitudes z , y , and x , respectively*: and therefore, (*art. 12.*) the Sum of the Fluxions of these pyramids will be equal to the Fluxion of the parallelopipedon xyz . Let either of the pyramids be represented by AEI or ACB , which suppose to be generated by the variable plane AF moving with a parallel motion along the indefinite right line AE ; and, at the same time, let the parallelopipedon $ADGE$ (whose face AD or EG is equal and similar to the base of the pyramid at the term CB) be generated by the plane AD always coincident with the plane AF . Now, it is plain, that, the pyramid will increase slower, or flow with a less degree of velocity, than the parallelopipedon, before the generating planes arrive at the term CB , and afterwards, faster, or with a greater degree of velocity; therefore, at the said term CB , they will flow, or be generated, with the same or an equal degree of velocity: but, it is likewise plain, that, the velocity with which the parallelopipedon is generated, is equal to it's face AD or CB drawn into the

Fig.
7.

* 7 E. 12. Corol. 1.

the velocity of it's motion along the line or side AE; therefore, the velocity with which the pyramid is generated, is, at the term CB, equal to it's base CB drawn into the velocity of it's motion, at the point C, along the side AE: that is, the Fluxion of the pyramid ACB is equal to it's base CB drawn into the Fluxion of it's altitude AC. Hence therefore, when the base is xy and altitude z , the Fluxion of the pyramid is $= xy \times \dot{z} = x y \dot{z}$; and when the base is xz and altitude y , the Fluxion of it is $= xz \times \dot{y} = x \dot{y} z$; and, lastly, when the base is yz and altitude x , it's Fluxion is $= yz \times \dot{x} = \dot{x} y z$. Consequently, $\dot{x} y z + x \dot{y} z + x y \dot{z}$ (which is the Sum of the Fluxions of the three pyramids,) is $=$ the Fluxion of the parallelepipedon xyz .

3^o. And, hence, that the Fluxion of $vxyz$ is $= \dot{v}xyz + v\dot{x}yz + vx\dot{y}z + vxy\dot{z}$, may be thus proved. Put $xyz = u$; then $vxyz = vu$: therefore, (*art.* 12.) the Fluxion of u is $=$ the Fluxion of xyz , and the Fluxion of vu is $=$ the Fluxion of $vxyz$; that is, $\dot{u} = \dot{x}yz + x\dot{y}z + xy\dot{z}$, and $\dot{v}u + v\dot{u} =$ the Fluxion of $vxyz$. Wherefore, by restitution, or writing xyz for u , and $\dot{x}yz + x\dot{y}z + xy\dot{z}$ for $\dot{u}$, we have

have $\dot{v}u + v\dot{u} = \dot{v}xyz + v\dot{x}yz + vx\dot{y}z + vx\dot{y}\dot{z} =$ the Fluxion of $vxyz$.

HENCE,

16. The Fluxion of xx is $= \dot{x}x + x\dot{x}$; the Fluxion of xxx is $= \dot{x}xx + x\dot{x}x + x\dot{x}x$; the Fluxion of $xxxx$ is $= \dot{x}xxx + x\dot{x}xx + x\dot{x}xx + x\dot{x}xx$: that is, the Fluxion of x^2 is $= 2x\dot{x}$; the Fluxion of x^3 is $= 3x^2\dot{x}$; the Fluxion of x^4 is $= 4x^3\dot{x}$. And, if m represents any affirmative or positive whole-number, the Fluxion of x^m will be $= mx^{m-1}\dot{x}$.

RULE III.

17. To find the Fluxion of a Fraction.

MULTIPLY the Denominator into the Fluxion of the Numerator; from the product of which, subtract the Numerator drawn into the Fluxion of the Denominator; then, divide the Remainder by the Square of the Denominator: and you will have the Fluxion of the Fraction required.

Thus, the Fluxion of $\frac{x}{y}$ is $= \frac{\dot{x}y - x\dot{y}}{y^2}$.

For, put $z = \frac{x}{y}$; then $yz = x$; and the Fluxion of this equation (*art.* 12 and 15.) is $\dot{y}z + y\dot{z} = \dot{x}$; therefore, by transposition,

D 2

$y\dot{z}$

$y\dot{z} = \dot{x} - y\dot{z}$, and, by division, $\dot{z} = \frac{\dot{x} - y\dot{z}}{y}$;

that is, by restitution, or writing $\frac{x}{y}$ for z it's

equal, $\dot{z} = \frac{\dot{x} - y \times \frac{x}{y}}{y} = \frac{\dot{x}y - x\dot{y}}{y^2} =$ the Fluxion of $\frac{x}{y}$.

Also, the Fluxion of $\frac{1}{x}$ is $= \frac{-\dot{x}}{x^2}$; the Fluxion of $\frac{1}{x^2}$ is $= \frac{-2x\dot{x}}{x^4}$; the Fluxion of

$\frac{1}{x^3}$ is $= \frac{-3x^2\dot{x}}{x^6}$; and the Fluxion of

$\frac{1}{x^4}$ is $= \frac{-4x^3\dot{x}}{x^8}$. For,

1^o. Put $z = \frac{1}{x}$; then $xz = 1$, and the Fluxion of xz will be $=$ the Fluxion of 1 , which (*art. 11.*) is $= 0$; that is, (*art. 15.*) $x\dot{z} + z\dot{x} = 0$: therefore, $x\dot{z} = -\dot{x}z$, and $\dot{z} = \frac{-\dot{x}z}{x}$; or, by writing for z it's equal $\frac{1}{x}$, $\dot{z} = \frac{-\dot{x}}{x^2} =$ the Fluxion of $\frac{1}{x}$.

2^o. Put $z = \frac{1}{x^2}$; then, $zx^2 = 1$; and the Fluxion of this equation (*art. 11. 15. and 16.*)

16.) is $\dot{z}x^2 + 2zx\dot{x} = 0$: therefore, $\dot{z}x^2 = -2zx\dot{x}$, and $\dot{z} = \frac{-2zx\dot{x}}{x^2}$; or, by writing $\frac{1}{x^2}$ for it's Value z , $\dot{z} = \frac{-2x\dot{x}}{x^4} = \frac{-2\dot{x}}{x^3} =$ the Fluxion of $\frac{1}{x^2}$.

3°. Put $z = \frac{1}{x^3}$; then, $zx^3 = 1$; and the Fluxion of this Equation (*art.* 11. 15. and 16.) is $\dot{z}x^3 + 3zx^2\dot{x} = 0$: therefore, $\dot{z}x^3 = -3zx^2\dot{x}$, and $\dot{z} = \frac{-3zx^2\dot{x}}{x^3}$; or, by writing $\frac{1}{x^3}$ for z it's value, $\dot{z} = \frac{-3x^2\dot{x}}{x^6} = \frac{-3\dot{x}}{x^4} =$ the Fluxion of $\frac{1}{x^3}$.

4°. Put $z = \frac{1}{x^4}$; then, $zx^4 = 1$; the Fluxion of which Equation (*art.* 11. 15. and 16.) is $\dot{z}x^4 + 4zx^3\dot{x} = 0$: therefore $\dot{z}x^4 = -4zx^3\dot{x}$, and $\dot{z} = \frac{-4zx^3\dot{x}}{x^4}$; that is, by restitution, or writing $\frac{1}{x^4}$ for z , $\dot{z} = \frac{-4x^3\dot{x}}{x^8} = \frac{-4\dot{x}}{x^5} =$ the Fluxion of $\frac{1}{x^4}$.

BUT,

BUT, by the *new* Method of Notation in Algebra, (*art.* 13.) $\frac{1}{x}$ is $= x^{-1}$, $\frac{1}{x^2}$ is $= x^{-2}$, $\frac{1}{x^3}$ is $= x^{-3}$, $\frac{1}{x^4}$ is $= x^{-4}$: also, $\frac{-\dot{x}}{x^2}$ is $= -x^{-2} \dot{x}$, $\frac{-2\dot{x}}{x^3}$ is $= -2 x^{-3} \dot{x}$, $\frac{-3\dot{x}}{x^4}$ is $= -3 x^{-4} \dot{x}$, $\frac{-4\dot{x}}{x^5}$ is $= -4 x^{-5} \dot{x}$.

HENCE,

18. The Fluxion of x^{-1} is $= -x^{-2} \dot{x}$, the Fluxion of x^{-2} is $= -2x^{-3} \dot{x}$, the Fluxion of x^{-3} is $= -3 x^{-4} \dot{x}$, the Fluxion of x^{-4} is $= -4 x^{-5} \dot{x}$. And, if m represents any negative whole-number, the Fluxion of x^m will be $= m x^{m-1} \dot{x}$.

RULE IV.

19. To find the Fluxion of an expression compounded of different terms or quantities connected together by the Signs $+$ and $-$.

FIND the Fluxion of each term by the preceding Rules; which connect together by the Signs of the respective terms: and you will have the Fluxion required.

Thus, the Fluxion of $ax + xy - \frac{x}{y}$ is $=$

$$a\dot{x} + \dot{x}y + x\dot{y} - \frac{\dot{x}y - x\dot{y}}{y^2}.$$

For,

For, put $ax + xy - \frac{x}{y} = v$; then, $ax + xy = v + \frac{x}{y}$. Now, (*art.* 12.) it is evident, that, the Sum of the Fluxions of ax and xy must always be equal to the Sum of the Fluxions of v and $\frac{x}{y}$; that is, $a\dot{x} + \dot{x}y + x\dot{y} = \dot{v} + \frac{\dot{x}y - x\dot{y}}{y^2}$: therefore, by transposition, $a\dot{x} + \dot{x}y + x\dot{y} - \frac{\dot{x}y - x\dot{y}}{y^2} = \dot{v}$
 $=$ the Fluxion of $ax + xy - \frac{x}{y}$.

RULE V.

20. To find the Fluxion of any Power of a given Fluent; whether the Index be integral or fractional, affirmative or negative.

MULTIPLY the expression by the Index of the Power; then, subtract 1 from the said Index, and multiply the resulting expression by the Fluxion of the given Fluent, or of the Root of the Power: and you will have the Fluxion required.

Thus, the Fluxion of $\overline{x + x^2}^3$ is $= 3 \times \overline{x + x^2}^{3-1} \times \dot{x} + 2x\dot{x} = \overline{x + x^2}^2 \times 3\dot{x} + 6x\dot{x}$;
the

24 *An Introduction to the*

the Fluxion of $\sqrt{2ax - x^2}$ is $= \frac{1}{2} \times$

$$\sqrt{2ax - x^2}^{\frac{1}{2}-1} \times 2a\dot{x} - 2x\dot{x} = \sqrt{2ax - x^2}^{-\frac{1}{2}}$$

$$\times a\dot{x} - x\dot{x} = (\text{because, art. 13. } \sqrt{2ax - x^2}^{-\frac{1}{2}})$$

$$\text{is } = \frac{1}{\sqrt{2ax - x^2}^{\frac{1}{2}}}, \frac{a\dot{x} - x\dot{x}}{\sqrt{2ax - x^2}^{\frac{1}{2}}}; \text{ the Flu-}$$

$$\text{xion of } \sqrt{a+x}^{-2} \text{ is } = -2 \times \sqrt{a+x}^{-2-1}$$

$$\times \dot{x} = \sqrt{a+x}^{-3} \times -2\dot{x} = \frac{-2\dot{x}}{\sqrt{a+x}^3}; \text{ the}$$

$$\text{Fluxion of } \sqrt{x-a}^{-\frac{2}{3}} \text{ is } = -\frac{2}{3} \times \sqrt{x-a}^{-\frac{2}{3}-1}$$

$$\times \dot{x} = -\frac{2}{3} \times \sqrt{x-a}^{-\frac{5}{3}} \times \dot{x} = \frac{1}{\sqrt{x-a}^{\frac{5}{3}}} \times -\frac{2\dot{x}}{3}$$

$$= \frac{-2\dot{x}}{3 \times \sqrt{x-a}^{\frac{5}{3}}}. \text{ And, Universally, the}$$

$$\text{Fluxion of } x^{\frac{m}{n}} \text{ is } = \frac{m}{n} \times x^{\frac{m}{n}-1} \times \dot{x} =$$

$$\frac{m}{n} x^{\frac{m}{n}-1} \dot{x} = \frac{m}{n} x^{\frac{m-n}{n}} \dot{x}; \text{ where, note, either}$$

m or n , or both m and n , may be either integral or fractional, affirmative or negative;

and consequently, the Index $\frac{m}{n}$, expresses,

or represents, any affirmative or negative whole-number or fraction whatsoever. For, by art. 16 and 18, if m represents any affirmative

mative or negative whole-number, the Fluxion of x^m will be $= m x^{m-1} \dot{x}$; and therefore, if we suppose $y = x^{\frac{m}{n}}$, or, which is the same, $y^n = x^m$, and n to be any affirmative or negative whole-number likewise, by *art.* 12. the Fluxion of y^n will be $=$ the Fluxion of x^m ; that is, $n y^{n-1} \dot{y} = m x^{m-1} \dot{x}$; which equation divided by $n y^{n-1}$ makes $\dot{y} = \frac{m x^{m-1} \dot{x}}{n y^{n-1}}$,

that is, by writing $x^{\frac{m}{n}}$ for y it's value, $\dot{y} = \frac{m x^{m-1} \dot{x}}{n \times x^{\frac{m}{n} \times n-1}} = \frac{m}{n} \times \frac{x^{m-1} \dot{x}}{x^{\frac{mn-m}{n}}} = \frac{m}{n} \times x^{m-1-\frac{mn-m}{n}} \dot{x}$
 $= \frac{m}{n} x^{\frac{m-n}{n}} \dot{x} =$ the Fluxion of $x^{\frac{m}{n}}$: and that

either m or n , or both m and n , may represent any fractions as well as whole-numbers, is plain, since $\frac{m}{n}$ represents the quotient of any whole-number divided by another, and may be taken for a New m or n ; and so on *ad infinitum*: therefore, Universally, &c.

RULE VI.

21. To find the Fluxion of a Logarithm.

THE Fluxion of the Hyperbolic Logarithm of any quantity, is equal to the Flu-

E xion

xion of that quantity, divided by the quantity itself.

Thus, the Fluxion of the Hyp. Log. of x is $= \frac{\dot{x}}{x}$. (See *Part 3. Quest. 8.*)

Now, as the Hyp. Log. of 10 (*viz.* 2.30258 &c. See *Part 3. Quest. 9.*) is to the Common Log. of 10 (*viz.* 1.) so is the Hyp. Log. of any number, x , to the Common Log. of the same number, x ; that is, if we put $L = 2.30258$ &c. as $L : 1 :: \text{Hyp. Log. of } x : \text{Common Log. of } x$. Therefore, (*art.* 12.) $L : 1 :: \text{Fluxion of the Hyp. Log. of } x : \text{Fluxion of the Common Log. of } x$.

HENCE, $L : 1 :: \frac{\dot{x}}{x} : \frac{\dot{x}}{Lx} =$ the Fluxion of the Common Log. of x ; or, because $\frac{1}{L} = 0.43429$ &c. if we put $\frac{1}{L} = M$, then the Fluxion of the Common Log. of x (*viz.* $\frac{\dot{x}}{Lx}$) will be $= \frac{\dot{x}}{x} \times M$; that is, the Fluxion of the Hyp. Log. of any number multiplied by (M or) 0.43429 &c. is = the Fluxion of the Common Log. of the said Number.

SCHO-

SCHOLIUM.

22. THOUGH hitherto we have supposed when one variable quantity in a Fluential * Expression increaseth, that, the others, if any, increase likewise: yet, it often happens, that some of them decrease while the others increase; in which case, the Fluxions of the decreasing are negative with respect to those of the increasing quantities: and therefore, the Signs of the terms affected with them ought to be changed.

Thus, if whilst x increases y decreases, the Fluxion of the rectangle xy will be expressed by $\dot{x}y - x\dot{y}$.

For, let the flowing or variable rectangle *Fig.*
 A B be continually increasing by the paral- 8.
 lel motion of the variable line C B moving
 from the situation A F along the line A E;
 and be continually diminishing by the mo-
 tion of the variable line D B, parallel to the
 line A E, and approaching towards it along
 the line F A; that is, let the side A C =
 D B continually increase whilst the Side A D
 = C B continually decreases. Now, put-
 ting A C = x , and A D = y , it follows
 from

* By a Fluential Expression is meant that which contains one, two, or more variable quantities.

from *art. 15. dem. 1^o*. that the Fluxion of the Increase will be $= \dot{x}y$, and that the Fluxion of the Decrease will be $= x\dot{y}$: therefore, the Fluxion of the Decrease subtracted from the Fluxion of the Increase, leaves the Fluxion of the rectangle $x y$ or the Velocity with which it flows $= \dot{x}y - x\dot{y}$.

If what has been said be well understood, it is hoped, the Learner will meet with few difficulties in the Application thereof; to which we now proceed.

C H A P. III.

Of drawing Tangents to Curves.

23. A *Tangent* is a right line which coincides with a Curve in a point, and there shews it's Direction, that is, the Inclination it bears to the Axis or the Angle it makes with the Ordinate.

Fig. Thus, if the right line T B coincides with
 9. the Curve A Y at any Point B, the said line
 10. is a Tangent to it at that Point: and, because no two lines can coincide unless they have the same direction, therefore, the direction

rection of the Tangent is properly the direction of the Curve at the point of contact.

Now, in general, what is requisite in order to draw the Tangent to any point B , is to find the right line CT , called the Subtangent; or, the distance of the point T from the ordinate CB through which the Tangent must pass. And to effect this,

24. Let cb be supposed parallel to the ordinate CB , and Bn equal and parallel to Cc : then, if the line cb be removed towards CB , in a parallel motion, until it coincides with it; the moment before it's coincidence, the triangle Bnb will be in it's evanescent State; or, which is the same thing, if the said two lines be separated from their coincidence, the very first moment of their separation produces the said triangle in it's nascent state; and in that moment, the line nb , terminated by the line Bn at one end and by the curve at the other, comes indefinitely near to touch the Tangent TB produced: consequently the triangles bnB and $BC T$ come then indefinitely near to similarity, and may be considered as similar: wherefore, by 4 E. 6. $bn : nB :: BC : CT$; that is, (putting the absciss $AC = x$, the ordinate $CB = y$, $Cc = Bn = x'$, and $bn = y'$),

y'

$$y' : x' :: y : \frac{x'y}{y'} = CT; \text{ or, (art. 7.) } \dot{y} :$$

$$\dot{x} :: y : \frac{\dot{x}y}{\dot{y}} = CT.$$

Or, Suppose the indefinite right line AZ to move with a parallel and uniform motion along the axis AX ; and, at the same time, a point to move from A along the said line AZ with such velocity as always to be in the curve AY : then, when the line AZ comes to the situation CB , if the point moving along thereon were to continue on with an uniform motion, and the same degree of velocity with which it arrives at B , it is evident it would move along the Tangent TB produced; and therefore, when the point A or C arrives at c , the point moving along the line AZ would arrive at m : and, because velocity is always as the space uniformly described in a given time, therefore Cc or Bn will be as the velocity with which the point A moves along the axis, nm as the velocity with which the point moves from B along the line AZ , and Bm as the velocity with which the point describes the curve at B or moves from B to m ; that is, Cc or Bn will be as the Fluxion of the absciss AC , nm as the Fluxion of the ordinate

nate CB , and Bm as the Fluxion of the curve at the point B : but the triangles $m n B$ and $B C T$ are similar; therefore, by 4 E. 6. $m n : n B :: B C : C T$; that is, the Fluxion of the ordinate : the Fluxion of the absciss :: the ordinate : the subtangent; or, (putting the absciss $AC = x$, and the ordinate $CB = y$,) $\dot{y} : \dot{x} :: y : \frac{\dot{x} y}{\dot{y}} = CT$; as before.

25. Now, $\frac{\dot{x} y}{\dot{y}}$ is a General Expression for the Subtangent of every curve whose absciss is x and ordinate y ; but, as it is embarrassed with the Fluxions of x and y , so the whole Business is to exterminate them: and to do this, put the Equation of the curve into Fluxions; from which, or from other properties of the curve, find the value of $\dot{x}$ in terms that are all affected with $\dot{y}$, or, of $\dot{y}$ in terms that are all affected with $\dot{x}$; then, if for $\dot{x}$ or $\dot{y}$ we substitute it's value thus found, in this general expression, *viz.* $\frac{\dot{x} y}{\dot{y}}$, we shall have the Subtangent CT in known terms, or free from Fluxions; by which the sought Tangent TB , to a given point in the curve, may be easily drawn.

26. *Note.* When the Fluxions of x and y , or of the absciss and ordinate, are Negative to each other, that is, when x increases if y decreases; or *vice versa*; then, the General Expression $(\frac{\dot{x}y}{\dot{y}})$ for the Subtangent will be $-\frac{\dot{x}y}{\dot{y}}$; wherein, the Negative sign only signifies, that, the subtangent lies on the other side of the ordinate with regard to the absciss x . But, in finding the Fluxion of the Equation of the curve, the Fluxions of x and y must not be considered as Negative to each other, if you would have the sought expression for the subtangent come out Affirmative.

*

EXAMPLE I.

27. To draw a *Tangent* to a *Circle* *.

Fig.

11. Put the radius EA or $ED = a$, absciss $AC = x$, and ordinate $CB = y$; then, $CD = 2a - x$. Now, by 35 E. 3. $AC \times CD = CB^2$, that is, $2ax - x^2 = y^2$; and this equation put into Fluxions is $2a\dot{x} - 2x\dot{x} = 2y\dot{y}$; which divided by $2a - 2x$, makes $\dot{x} =$

* By the word *Circle*, is generally meant the *Space* bounded by a Curve every-where equidistant from a fixed point; but sometimes the *Curve* itself. (See Sir Isaac Newton's *Arithmetica Universalis*.)

$\dot{x} = \frac{y \dot{y}}{a-x}$; which substituted for $\dot{x}$ in the general expression for the Subtangent, (*viz.* $\frac{\dot{x} y}{\dot{y}}$, *art.* 25.) makes the Subtangent $CT = \frac{y^2}{a-x}$ (which, by writing $2ax - x^2$ for it's above value, *viz.* y^2 , is) $= \frac{2ax - x^2}{a-x} = \frac{CB^2}{EC}$. Wherefore, if the distance signified by this expression be set off from the point C, in the diameter DA produced, we shall have the point T through which the Tangent to the point B must pass.

Construction.

Through the point B describe the semi-circle EBT: then will T be the point from which the Tangent to the point B is to be drawn. For, by 31 E. 3. the angle EBT will be right; and therefore, by 8 E. 6. the triangles ECB and BCT will be similar, and by 4 E. 6. $EC : CB :: BC : CT$; $\therefore CT = \frac{CB^2}{EC}$.

F

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EXAMPLE II.

Fig. 28. To draw a *Tangent* to a *Parabola*.

12. Suppose F to be the focus; and PR the parameter, which put $= a$; also, put the absciss $AC = x$, and ordinate $CB = y$. Now, by a well known property of the curve, $PR \times AC = CB^2$, that is, $ax = y^2$; the Fluxion of which equation is $a\dot{x} = 2y\dot{y}$; therefore, $\dot{x} = \frac{2y\dot{y}}{a}$; which substituted for $\dot{x}$, makes the general expression for the Subtangent CT (*viz.* $\frac{\dot{x}y}{\dot{y}}$ *art.* 25.) $= \frac{2y^2}{a} =$
(by substituting ax for y^2 it's value,) $\frac{2ax}{a}$
 $= 2x$. So that, the Subtangent CT is double the absciss AC; and consequently, AT is $= AC$.

EXAMPLE III.

Fig. 29. To draw a *Tangent* to an *Ellipsis*.

13. Put the transverse diameter $AD = a$, conjugate $NO = b$, absciss $AC = x$, and ordinate $CB = y$. Now, by a well known property of the curve, $AD^2 : NO^2 :: AC \times CD : CB^2$; that is, $a^2 : b^2 :: x \times \overline{a-x} :$
 $\frac{b^2}{a^2}$

$\frac{b^2}{a^2} \times \overline{ax - x^2} = y^2$; the Fluxion of which equation is $\frac{b^2}{a^2} \times \overline{a\dot{x} - 2x\dot{x}} = 2y\dot{y}$; and

this divided by $\frac{b^2}{a^2} \times \overline{a - 2x}$, gives $\dot{x} =$

$\frac{2a^2y\dot{y}}{ab^2 - 2b^2x}$; which substituted for $\dot{x}$ in $\frac{\dot{x}y}{\dot{y}}$

(the general expression for the Subtangent, *art.* 25.) makes the Subtangent CT =

$\frac{2a^2y^2}{ab^2 - 2b^2x} =$ (by writing $\frac{b^2}{a^2} \times \overline{ax - x^2}$

for y^2 it's equal,) $\frac{2a^3b^2x - 2a^2b^2x^2}{a^3b^2 - 2a^2b^2x} =$

$\frac{2ax - 2x^2}{a - 2x}$. Whence we may observe, that,

AT is ($= CT - CA = \frac{2ax - 2x^2}{a - 2x} - x$)

$= \frac{ax}{a - 2x} =$ that Part of the Subtangent

which falls *without* the curve.

Construction.

Make $Cv = CA$; erect the perpendicular Ar , terminated by a right line drawn from E through v ; lastly, make $AT = Ar$: then will T be the point from which the Tangent to the Point B must be drawn.

For, by 4 E. 6. $EC : Cv :: EA : Ar$, that

$$\text{is, } \frac{1}{2} a - x : x :: \frac{1}{2} a : \frac{ax}{a - 2x} = AT.$$

EXAMPLE IV.

Fig. 30. To draw a *Tangent* to an *Hyperbola*.
14.

Put the transverse diameter $DA = a$, conjugate $NO = b$, absciss $AC = x$, and ordinate $CB = y$. Now, by a well known property of the curve, $DA^2 : NO^2 :: DC \times AC : CB^2$, that is, $a^2 : b^2 :: a + x \times x : \frac{b^2}{a^2} \times ax + x^2 = y^2$; and, by putting both sides of this equation of the curve into Fluxions, we shall have $\frac{b^2}{a^2} \times a\dot{x} + 2x\dot{x} = 2y\dot{y}$; therefore, by division, $\dot{x} = \frac{2a^2y\dot{y}}{ab^2 + 2b^2x}$; which multiplied by $\frac{y}{\dot{y}}$, or, which is the same, substituted for $\dot{x}$ in $\frac{\dot{x}y}{\dot{y}}$, (the general expression for the Subtangent, *art.* 25.) makes the Subtangent $CT = \frac{2a^2y^2}{ab^2 + 2b^2x}$ (which, by writing for y^2 it's equal $\frac{b^2}{a^2} \times \frac{ax + x^2}{a^2}$

$\overline{ax + x^2}$, is) $= \frac{2ax + 2x^2}{a + 2x}$. So that,
 AT, that Part of the Subtangent *without* the
 curve, is $= \frac{2ax + 2x^2}{a + 2x} - x = \frac{ax}{a + 2x}$.

Construction.

Make $Cv = CA$; erect the perpendicular
 Ar, terminated by the right line Ev; lastly,
 make $AT = Ar$: then will T be the point
 from which the Tangent to the point B must
 be drawn. For, by 4E. 6. $EC : Cv :: EA :$
 Ar , that is, $\frac{1}{2}a + x : x :: \frac{1}{2}a : \frac{ax}{a + 2x} =$
 AT,

SCHOLIUM.

31. FROM the three foregoing Examples,
 it may be observed, that, in the *Parabola* (fig.
 12.) the internal part of the Subtangent, *viz.*
 the absciss AC, is always *equal* to the exter-
 nal part AT: In the *Ellipsis* (fig. 13.) the
 internal part AC is always *less* than the ex-
 ternal part AT: And, that, in the *Hyper-*
bola (fig. 14.) the internal part AC is always
greater than the external part AT.

Ex-

EXAMPLE V.

Fig. 32. To draw a *Tangent* to an *Hyperbola* between it's *Asymptotes*; that is, taking one of it's *Asymptotes* for an *Axis*.
15.

Let EZ and ET be the asymptotes of the hyperbola YAB , whose vertex is A ; draw AP and BC parallel to the asymptote EZ ; then will AP be the parameter and equal to PE , which put $= a$; EC an absciss, which put $= x$; and CB an ordinate, which put $= y$. Now, because when x increases y decreases; therefore $\dot{x}$ and $\dot{y}$ are negative to each other, and the general expression for the Subtangent is $-\frac{\dot{x}y}{\dot{y}}$; where

the negative sign shews, that, the point T lies on the other side of the ordinate CB with regard to E , (*art.* 26.). By the well known property of the curve, $EC : EP ::$

$PA : CB$, that is, $x : a :: a : y$; $\therefore x = \frac{a^2}{y}$;

the Fluxion of which equation is $\dot{x} = -\frac{a^2\dot{y}}{y^2}$;

and this value of $\dot{x}$ being substituted for it in the above general expression for the Subtangent,

tangent, viz: $-\frac{\dot{x}y}{\dot{y}}$, makes the Subtangent
 $CT = \frac{a^2 y \dot{y}}{y^2 \dot{y}} = \frac{a^2}{y}$ (by writing $x y$ for it's
 equal a^2), $\frac{x y}{y} = x$. So that, CT must be
 $= CE$.

EXAMPLE VI.

33. To draw a *Tangent* to the *Conchoid* of *Fig.*
Nicomedes *. 16.

Let fall the perpendicular BH on the
 asymptote EZ , and draw BC equal and pa-
 rallel to HE . Put $PE = a$, $EA = b = FB$,
 $EC = x = HB$, and $CB = y = EH$. Now,
 by 47 E. 1. $\sqrt{FB^2 - BH^2}^{\frac{1}{2}} = HF$, that is,
 $\sqrt{b^2 - x^2}^{\frac{1}{2}} = HF$; and, because the triangles
 PCB and BHF are similar, by 4 E. 6.
 $BH : HF :: PC : CB$, that is, $x : \sqrt{b^2 - x^2}^{\frac{1}{2}}$
 $:: a + x : y$; $\therefore y = \frac{a + x}{x} \times \sqrt{b^2 - x^2}^{\frac{1}{2}}$; which
 is

* This Curve is thus generated.—From a fixed Point P , which is called the Pole of the Conchoid, let a number of right lines PA, PB, PD , &c. be drawn, cutting the right line EZ , which is an asymptote to the Curve; and let the distances EA, FB, GD , &c. be made equal to each other, and a line be drawn through the points A, B, D , &c. then will this line be a curve, called by it's Inventor *Nicomedes* a *Conchoid*.

is the equation of the curve; the Fluxion of

$$\text{which is } \dot{y} = \frac{x \dot{x} - a + x \dot{x}}{x^2} \times b^2 - x^2 \Big|^{\frac{1}{2}} - \frac{1}{2}$$

$$\times b^2 - x^2 \Big|^{-\frac{1}{2}} 2 x \dot{x} \times \frac{a + x}{x} = \frac{-a \dot{x}}{x^2} \times b^2 - x^2 \Big|^{\frac{1}{2}}$$

$$- \frac{x \dot{x}}{b^2 - x^2 \Big|^{\frac{1}{2}}} \times \frac{a + x}{x} = \frac{-ab^2 \dot{x} - x^3 \dot{x}}{x^2 \times b^2 - x^2 \Big|^{\frac{1}{2}}}; \text{ which}$$

substituted for $\dot{y}$, in $-\frac{\dot{x}y}{\dot{y}}$, the general expression for the Subtangent when $\dot{x}$ and $\dot{y}$ are negative to each other, (*art. 26.*) makes the

$$\text{Subtangent CT} = \frac{y x^2 \times b^2 - x^2 \Big|^{\frac{1}{2}}}{ab^2 + x^3} \quad (\text{which,}$$

$$\text{by substituting for } y \text{ it's equal in the above equation of the curve, is)} = \frac{a + x \times b^2 - x^2 \times x^2}{ab^2 x + x^4}$$

$$= \frac{a + x \times b^2 x - x^3}{ab^2 + x^3}.$$

Construction.

Make $Bn = PC$, draw nr parallel to CP , make $Fv = nr$, draw the right line vC , parallel to which draw BT : then will BT be a Tangent to the point B . For, by 4 E. 6. $PC : CB :: PE : EF$, that is $a + x : y :: a :$

$$\frac{a y}{a + x}$$

$\frac{ay}{a+x} = EF$; and $BC : CP :: Bn : nr$, that

is, $y : a+x :: a+x : \frac{(a+x)^2}{y} = nr = Fv$;

$\therefore Ev = (EF + Fv =) \frac{ay}{a+x} + \frac{(a+x)^2}{y}$.

Again, $vE : EC :: BC : CT$, that is, $\frac{ay}{a+x}$

$+ \frac{(a+x)^2}{y} : x :: y : \frac{xy^2 \times (a+x)}{ay^2 + (a+x)^3} = CT$. But,

by the equation of the curve, $y^2 = \frac{(a+x)^2}{x^2} \times$

$\overline{b^2 - x^2}$; which substituted for y^2 , makes CT

$$= \frac{a+x \times \overline{b^2 x - x^3}}{ab^2 + x^3}.$$

EXAMPLE VII.

34. To draw a *Tangent* to the *Cissoïd* of *Fig.*
Diocles *. 17.

Let ABD be the *Cissoïd*, whose generating
femicircle is AfE , and asymptote EZ an in-
definite

* This Curve is thus generated.—In the semicircle AfE , make any two arches Ea and Ac , or Eb and Ad , equal to one another; and through the points a, b, d, c , let right lines be drawn perpendicular to the diameter AE , and transverse lines from the point A ; then, from A , through the points of intersection O, B, D , &c. draw a line $A \odot BD$ &c. and it will be a Curve, called by it's Inventor *Diocles* a *Cissoïd*.

definite right line perpendicular to the diameter AE. Put the diameter $AE = a$, absciss $AC = x$, ordinate $CB = y$. Now, because by the generation of the curve, the arches Eb and Ad are equal, therefore $bb = dC$, and $bE = CA$; and, because the triangles bbA and BCA are alike, therefore, by 4 E. 6. $Ab : bb :: AC : CB$; but, by 13 E. 6. $Ab : bb :: bb : bE$; $\therefore bb : bE :: AC : CB$; that is, $dC : CA :: AC : CB$; or (because $CE = a - x$, and by 35 E. 3. $Cd = \overline{AC \times CE}^{\frac{1}{2}} = \overline{ax - x^2}^{\frac{1}{2}}$), $\overline{ax - x^2}^{\frac{1}{2}} : x :: x : y$; $\therefore x^2 = y \times \overline{ax - x^2}^{\frac{1}{2}}$; the square of which equation divided by x , is $x^3 = ay^2 - xy^2$; which is the equation of the curve; and the Fluxion of this equation is $3x^2\dot{x} = 2ay\dot{y} - \dot{x}y^2 - 2xy\dot{y}$; therefore, by transposition and division, $\dot{x} = \frac{2ay\dot{y} - 2xy\dot{y}}{3x^2 + y^2}$, which substituted for $\dot{x}$, makes the Subtangent CT ($= \frac{\dot{x}y}{\dot{y}}$, art. 25.) = $\frac{2ay^2 - 2xy^2}{3x^2 + y^2}$ = (by writing for y^2 it's equal $\frac{x^3}{a-x}$), $\frac{2ax - 2x^2}{3a - 2x}$. Whence we may observe, that, AT, the Difference between the absciss AC and subtangent TC, is = $\frac{ax}{3a - 2x}$.

Con-

Construction.

Bisect the radius eE in g , and the absciss AC in f ; draw the perpendicular Ar equal to fg ; make $rn = Af$; and on the point n erect the perpendicular nv , terminated by the right line er ; lastly, draw vT equal and parallel to nA : then will T be the point from which the Tangent to the point B must be drawn. For, $rn = Af = \frac{1}{2}x$, and $rA = Ag - Af = \frac{3}{4}a - \frac{1}{2}x$; and by 4 E. 6. $rA : Ae :: rn : nv$ or AT ; that is, $\frac{3}{4}a - \frac{1}{2}x : \frac{1}{2}a :: \frac{1}{2}x : \frac{ax}{3a - 2x} = AT$.

EXAMPLE VIII.

35. To draw a *Tangent* to the *Cycloid* *. Fig. 18.

Put OA the radius of the generating circle $= a$, absciss $AC = x$, ordinate $CB = y$, $CG = s$, and the arch $AG = z$. Now, by the nature of the curve, $CB = CG + \text{arch } GA$; that is, $y = s + z$; (for, when the semicircle AGF

* This Curve is thus generated.—Let a circle roll along upon a right line until it performs one revolution, that is, until it measures out a right line equal to it's circumference; then, that point in the circle which first touched the right line will describe the curve called a *Cycloid*; which curve, it is supposed, was first invented by Cardinal *Cusanus*; whose works, in which this figure is found, were transcribed by *J. Scoblar* in the year 1451. — He (*Nicolas de Cusa*) was born in 1401, made Cardinal of Rome in 1448, and died in 1464.

AGF, generating the femicycloid ABD, is in the position BRK, the arch BR or GF must evidently be equal to RD, and the arch RK or BM or GA be equal to RF or tC ; but CG is $= tB$, and therefore $Ct = GB$: consequently, arch $GA = GB$; and therefore, &c.) and the Fluxion of this equation of the curve is $\dot{y} = \dot{s} + \dot{z}$. Let cg be supposed indefinitely near and parallel to CG, and Gn equal and parallel to Cc ; that is, let $Gg = z'$, $gn = s'$, and $nG = Cc = x'$: then, supposing Gg to be a little right line perpendicular to the radius GO, the angles gGn and OGC will be equal; for, if to either of them the angle OGn be added, the sum will be a right angle; and therefore, the right angled triangles gnG and OCG are alike: consequently, by 4 E. 6. $gn : nG :: OC : CG$; that is,

$$s' : x' :: a - x : s; \therefore s' = \frac{a-x}{s} x', \text{ or (art.}$$

$$7.) \dot{s} = \frac{a-x}{s} \dot{x}; \text{ and, } gG : Gn :: OG : GC;$$

$$\text{that is, } z' : x' :: a : s; \therefore z' = \frac{a}{s} x', \text{ or, } \dot{z}$$

$$= \frac{a}{s} \dot{x}. \text{ Now, by substituting } \frac{a-x}{s} \dot{x} \text{ for } \dot{s},$$

$$\text{and } \frac{a}{s} \dot{x} \text{ for } \dot{z}, \text{ in the above Fluxion of the}$$

equa-

equation of the curve, we have $\dot{y} = \frac{a-x}{s} \dot{x}$
 $+ \frac{a}{s} \dot{x} = \frac{2a-x}{s} \dot{x}$; and this substituted for
 $\dot{y}$, makes the Subtangent CT ($= \frac{\dot{x}y}{\dot{y}}$, *art.*
 25.) $= \frac{sy}{2a-x}$.

Construction.

Draw the chord or right line AG, and parallel to it draw TB: then will TB be a Tangent to the curve at the point B. For, the triangles FCG and BCT are similar; therefore, by 4 E. 6. FC : CG :: BC : CT, that is, $2a-x : s :: y : \frac{sy}{2a-x} = CT$.

Or,

Draw the right line EG perpendicular to the radius GO and equal to GB; and through the point E draw the right line TB: then will the said line TB be a Tangent to the curve at the point B. For, let gb be supposed indefinitely near and parallel to GB; then, because the arch AG = GB = gv , and the arch AG + Gg = $gv + vb$; therefore Gg or Bv = vb ; consequently,
 if

if we suppose Gg to be a little right line perpendicular to the radius OG or coinciding with the right line EG produced, and Bv to be a little right line parallel to it, and also Bb to be a little right line coinciding with a Tangent to the curve at the point B , the triangles bvB and BGE will be similar: therefore, &c.

EXAMPLE IX.

Fig. 36. To draw a *Tangent* to the Curve AB ,
 19. whose Equation (putting $GC=x$, $CB=y$,
 and $a =$ a given quantity *more* than 1,) is $a^x = y$.

Put $A =$ the Hyperbolic Logarithm of a ,
 and $Y =$ the Hyperbolic Logarithm of y ;
 then, by the nature of Logarithms, $x A = Y$;
 the Fluxion of which equation is $\dot{x} A = \dot{Y} =$
 (by *art. 21*) $\frac{\dot{y}}{y}$; which divided by A , makes
 $\dot{x} = \frac{\dot{y}}{A y}$; and this substituted for $\dot{x}$, makes
 the Subtangent $CT (= \frac{\dot{x} y}{\dot{y}}, \text{art. 25.}) = \frac{y}{A y}$
 $= \frac{1}{A}$. Whence we may observe, that, the
 Subtangent being an invariable quantity, the
 Curve

Curve AB is the *Logarithmic Curve*, whose Asymptote is GF *.

37. HITHERTO we have treated only of Curves referred to an Axis; or, of those whose ordinates are Parallel to one another: We shall therefore now proceed to the drawing of Tangents to *Spirals*; or, to those Curves whose ordinates issue from one and the same fixed Point: Where *note*, the ordinate and subtangent are always Perpendicular to each other:

38. SUPPOSE Cb indefinitely near to CB; *Fig.* that is, let the $\angle BCb$ be supposed indefinitely ^{20.} small; and with the ordinate CB, as a radius, let the indefinitely small arch Bn be described; which being considered as a little right line perpendicular to Cb , and the indefinitely small part of the curve Bb as coinciding with a Tangent to it at the point B; then, because the $\angle BCb$ is indefinitely small,

* This Curve is thus generated.—In the indefinite right line GF, make $GK=KC=CF$, &c. and the perpendiculars GA, KI, CB, FD, &c. in geometrical proportion continued: then, a curve drawn through the points A, I, B, D, &c. will be the *Logarithmic Curve*; which is so called, because the distances GK, KC, CF, &c. being in arithmetical progression, are as the Logarithms of the ordinates KI, CB, FD, &c.

small, the $\triangle s\ bnB$ and BCT will be indefinitely near to similarity; that is, in the very first moment of the existence of the $\triangle bnB$, the said $\triangle$ may be considered as similar to the $\triangle BCT$; therefore, if we put the ordinate $CB=y$, $Bn=x'$, and $nb=y'$, by 4 E. 6. we shall have $y' : x' :: y : CT$; that is,

$$(art. 7.) \dot{y} : \dot{x} :: y : CT; \therefore CT = \frac{\dot{x}y}{\dot{y}};$$

which is the same General Expression for the Subtangent as that before found for curves referred to an Axis, *art. 25.*

Or, Let the indefinite right line CZ turn, like the radius of a circle, round the fixed point or center C with an uniform motion; and, at the same time, let a point moving along thereon, generate, or move with such degrees of velocity as always to be in the Curve CBY , to which suppose the right line TG a Tangent at the point B ; also, let Bm be a Tangent to the circular arch DBn described with the ordinate CB as a radius. Now, when the point generating the curve CY arrives at B , if it was to continue on in the same direction, with an uniform motion, and the same degree of velocity that it arrives thereat, it would move along the Tangent

gent TB produced, or right line BG; which right line would always be as the Fluxion of the Spiral at the point B: So likewise, if we suppose a point to move from B, in the same direction, and with the same uniform motion, that the point generating the circular arch DB n arrives at B, it would move along the Tangent or right line B m perpendicular to the ordinate or radius CB; which right line would always be as the Fluxion of the said arch at the point B: And, since the direction of the point moving from C to Z is always perpendicular to that of the point generating the circular arch D n ; therefore, when the point moving from B to G arrives at G, if G m be parallel to ZC, the point moving along B m will be arrived at m ; and therefore, m G will be as the Fluxion of the ordinate at the point B. Hence, because the Δ s G m B and BC Γ are similar, by 4 E. 6. G m : m B :: BC : CT; that is, the Fluxion of the ordinate CB : the Fluxion of the arch DB :: the ordinate CB : the subtangent CT; or, (putting the arch DB= x , and the ordinate CB= y ,) $\dot{x}$: y :: y : $\frac{\dot{x}y}{\dot{y}}$ = CT; as before.

EXAMPLE I.

Fig. 39. To draw a *Tangent* to the *Spiral* of Archimedes *.

Put the circumference of the generating circle $AFA = a$, and it's radius $CA = b$; ordinate $CB = y$, arch $ARF = z$; and with the ordinate CB , as a radius, let the circular arch Bn be described, which put $= x$. Now, by the generation of the curve, $a : b :: z : y$, or $a : b :: \dot{z} : \dot{y}$, $\therefore \dot{z} = \frac{a\dot{y}}{b}$. But, it is

evident, the velocity of the point generating the circle is to the velocity with which the point B generates the arch Bn as CF is to CB ; that is, $\dot{z} : \dot{x} :: b : y$; $\therefore \dot{z} = \frac{b\dot{x}}{y}$. Hence

we have $\frac{b\dot{x}}{y} = \frac{a\dot{y}}{b}$; therefore, $b^2\dot{x} = ay\dot{y}$, and

$\dot{x} =$

* This Curve is thus generated.—With the radius CA , let the circle AFA be described with an equable motion, or the point A describe equal arches in equal times; and, at the same moment of time that the point A begins to generate the circle, let another point begin to move along the said radius from C towards A , and to pass over it with an uniform motion, and such velocity that it may arrive at A at the very same moment of time that the said radius shall have described the circle, or come to be in it's first situation: then will the point moving along the radius CA , generate, or describe, the curve CBA , called a *Spiral*; the Invention of which is attributed to Archimedes.

$\dot{x} = \frac{ay\dot{y}}{b^2}$, which substituted for $\dot{x}$ in the general expreffion for the Subtangent, $\frac{\dot{x}y}{\dot{y}}$, (*art.* 38.) makes the Subtangent $CT = \frac{ay^2}{b^2} =$
 (because $a : b :: z : y$, or $ay = bz$,) $\frac{byz}{b^2} = \frac{yz}{b}$.

Construction.

With the ordinate CB, as a radius, describe the circular arch BD; and draw the right line CT perpendicular to the radius CF and equal to the arch DB: then will T be the point from which the Tangent to the point B must be drawn. For, the sectors CFRA and CBD are fimilar; and confequently, $CF : FA :: CB : BD$; that is, $b : z :: y : BD = \frac{yz}{b} = CT$.

EXAMPLE II.

40. To draw a *Tangent* to the *Logarithmic Fig.*
Spiral *. 22.

Put the curve $CdB = z$, and it's correspondent ordinate $CB = y$. Suppose the
 angles

* This Curve is thus generated.—In the circle AFA, whose center is C, let any arches AD, DE, EF, FG, GH, &c.
 be

angles eCB and BCb to be indefinitely small and equal; and with the ordinate CB , as a radius, describe the circular arch Bn . Now, by the generation of the curve, $Ce : CB :: CB : Cb$; and therefore, if we consider the little parts of the curve, eB and Bb , as indefinitely little right lines, by 6 E. 6. the triangles CeB and CBb will be similar, or the angle made by the curve and ordinate be always the same. Consequently, the Increments of the ordinate and curve are always in an invariable ratio to each other, or as two fixed quantities a and b ; that is, $nb : bB :: a : b$; and therefore, if we suppose Bn to be a little right line perpendicular to Cb , by 47 E. 1. $bn : nB :: a : \sqrt{b^2 - a^2}^{\frac{1}{2}}$; that is, (putting $Bn = x'$, and $nb = y'$,) $y' : x' :: a : \sqrt{b^2 - a^2}^{\frac{1}{2}}$, or, (*art.* 7.) $\dot{y} : \dot{x} :: a : \sqrt{b^2 - a^2}^{\frac{1}{2}}$;
∴

be made equal to each other; and the right lines Cd , Ce , CB , Cb , &c. in geometrical proportion continued: then, a line drawn through the points d , b , B , e , d , &c. will be the curve called a *Logarithmic Spiral*; which Name is given to it, because the Arches AD , AE , AF , &c. being in arithmetical progression, are as the Logarithms of the ordinates Cd , Ce , CB , &c.

Scholium.

As no geometrical series can, strictly speaking, terminate in 0: so the *Spiral* Bed &c. though it continually tends towards the center C , can never absolutely arrive thereat; but will approach it within any distance that can be assigned.

$\therefore \dot{x} = \dot{y} \times \frac{\sqrt{b^2 - a^2}^{\frac{1}{2}}}{a}$; which being substituted for $\dot{x}$ in the general expression for the Subtangent, viz. $\frac{\dot{xy}}{\dot{y}}$, art. 38. makes the sought

$$\text{Subtangent CT} = y \times \frac{\sqrt{b^2 - a^2}^{\frac{1}{2}}}{a}.$$

Scholium.

We may observe, that, (because by 47 E. I. $\text{TB}^2 = \text{BC}^2 + \text{CT}^2 = y^2 + \frac{b^2 - a^2}{a^2} y^2 = \frac{b^2 y^2}{a^2}$,) the Tangent TB is $= y \times \frac{b}{a}$; and consequently, the Tangent and Subtangent are to each other as b to $\sqrt{b^2 - a^2}^{\frac{1}{2}}$. Again, because $nb : bB :: a : b$, that is, (art. 7.) $\dot{y} : \dot{z} :: a : b$; therefore, (art. 12.) $y : z :: a : b$; or, the ordinate and curve are always in a given or fixed ratio to each other; and therefore $z = y \times \frac{b}{a}$. So that, the Tangent TB and Curve CdB are equal; and sine $\angle \text{BTC} : \text{radius} :: a : b$, or, s. $\angle \text{BTC} = a$, and rad. (s. $\angle \text{BCT}$) $= b$.

Ex:

EXAMPLE III*.

Fig. 41. To draw a *Tangent* to the *Spiral* CHBLA,
 23. generated by a point moving uniformly
 24. along the semicircle CDA, from C to A,
 while the said semicircle makes one uni-
 form revolution round the point C as a
 center.

Let the point b be supposed indefinitely
 near to B; and with the ordinate CB, as a
 radius, describe the arch DB π . Put the ra-
 dius of the generating semicircle, EB, = a ,
 arch CRB = v , arch DB = x , ordinate
 CB = y , Bb = v' , and $b\pi$ = y' . Now,
 because the circumferences of circles are as
 their radii, by the generation of the curve

$$a : 2y :: \dot{v} : \dot{x} = \frac{2y\dot{v}}{a}; \text{ and, because by 20}$$

$$\text{E. 3. } \angle bEB = 2\angle BCb, \quad a : y :: v' : 2B\pi,$$

$$\therefore B\pi = \frac{yv'}{2a}; \text{ but, by 47 E. I. (supposing Bb}$$

to be a little right line, and B π a little right
 line perpendicular to Cb,) $Bb^2 - b\pi^2 = nB^2$,

$$\text{that is, } v'^2 - y'^2 = \frac{y^2 v'^2}{4a^2}; \text{ from which equa-}$$

$$\text{tion we have } v' = \frac{2ay'}{4a^2 - y'^2}^{\frac{1}{2}}, \text{ that is, (art.}$$

7.)

* Invented Anno 1756.

$$7.) \dot{v} = \frac{2a\dot{y}}{4a^2 - y^2|^{\frac{1}{2}}}. \quad \text{Hence, } \dot{x} = \frac{2y}{a} \times$$

$$\frac{2a\dot{y}}{4a^2 - y^2|^{\frac{1}{2}}} = \frac{4y\dot{y}}{4a^2 - y^2|^{\frac{1}{2}}}; \quad \text{which substituted}$$

for $\dot{x}$, makes $\frac{\dot{x}y}{\dot{y}}$, the general expression for

$$\text{the Subtangent CT (art. 38.)} = \frac{y}{\dot{y}} \times \frac{4y\dot{y}}{4a^2 - y^2|^{\frac{1}{2}}}$$

$$= \frac{4y^2}{4a^2 - y^2|^{\frac{1}{2}}}.$$

Corollary.

$$\text{Since } \dot{x} = \frac{2y\dot{v}}{a}, \text{ that is, } x' = \frac{2yv'}{a}; \text{ and } Bn$$

$$= \frac{yv'}{2a}: \text{ therefore, } x' = 4Bn.$$

Construction.

Draw the chord BG, which bisect in F;
and to the point F draw the right line CF;
produce the chord or ordinate CB to I, mak-
ing IB = BC; also, draw the right line IT,
making the angle CIT = the angle CFB:
then will T be the point from which the
Tangent to the point B must be drawn.
For, by 31 E. 3. the angle CBG is right, and
therefore the right angled triangles FBC and
ICT are similar; wherefore, by 4 E. 6. BF :
BC

BC :: IC : CT; that is, (because by 47 E. 1.
 $BG = \sqrt{GC^2 - CB^2} = \sqrt{4a^2 - y^2}$, and there-
 fore $BF = \frac{1}{2} \cdot \sqrt{4a^2 - y^2}$), $\frac{1}{2} \cdot \sqrt{4a^2 - y^2} : y ::$
 $2y : \frac{4y^2}{\sqrt{4a^2 - y^2}} = CT$.

C H A P. IV.

*Of finding the Maxima and Minima, or
 Greatest and Least of variable Quantities.*

42. A *Maximum* is the greatest magni-
 tude of a variable Quantity, which naturally
 increases before it arrives at that magnitude
 and as naturally decreases afterwards; and a
Minimum is the least magnitude of a variable
 Quantity, which naturally decreases before it
 arrives at that magnitude and afterwards as
 naturally increases. Therefore, at that cer-
 tain term where a Quantity becomes a Maxi-
 mum or a Minimum it can neither be in-
 creasing nor decreasing; and consequently,
 it's Fluxion is = 0.

Fig. Thus, in the triangle AEF, let the equal
 25. and parallel sides AC and DB of the inscribed
 and flowing rectangle CD be continually in-
 creasing while the other equal and parallel
 sides AD and CB are continually decreasing;
 that

that is, let the said rectangle be increased by the motion of the variable and decreasing side CB, and decreased by the motion of the variable and increasing side DB. Then, it is evident, that, while the rectangle is increasing faster, or with a greater degree of velocity, by the motion of the line CB, than it is decreasing by the motion of the line DB, it will be continually Increasing; and that, when it is decreasing faster, or with a greater degree of velocity, by the motion of the line DB, than it is increasing by the motion of the line CB, it will be continually Decreasing: therefore, when these two degrees of velocity become equal, the rectangle will be a Maximum; and be neither increasing nor decreasing; or, the Velocity or Fluxion with which it is decreasing subtracted from the Velocity or Fluxion with which it is increasing, leaves the Velocity or Fluxion with which it flows $= 0$. And, it being manifest, that, when the rectangle CD becomes a Maximum, the sum of the triangles FDB and BCE will be a Minimum; therefore, when the sum of the said triangles is a Minimum, the Fluxion of the increasing triangle FDB will be equal to the Fluxion of the decreasing triangle ECB; and therefore, the Flu-

xion of the latter subtracted from that of the former, gives the Fluxion of the Minimum = 0.

43. WHEN any Quantity is a Maximum or a Minimum, all the Affirmative Powers of it will be so too; as will also, the Sum, Remainder, Product or Quotient, arising from it's being added to, subtracted from, multiplied or divided by, any invariable or given quantity. But, any Negative * Power of a Maximum will be a Minimum; and any Negative Power of a Minimum will be a Maximum.

Thus, (1°..) if $a - b + \frac{c}{d} \times \overline{ax - x^2}^{\frac{1}{2}}$ be a Maximum, then will $ax - x^2$ be also a Maximum. And, (2°..) if $\overline{\frac{a^2 c^2}{x^2} + a^2 + c^2 + x^2}^{-1}$ be a Maximum, then will $\frac{a^2 c^2}{x^2} + a^2 + c^2 + x^2$ or $\frac{a^2 c^2}{x^2} + x^2$ be a Minimum.

For (1°..) it is evident, that, the greater $ax - x^2$ is, the greater will be $a - b + \frac{c}{d} \times \overline{ax - x^2}^{\frac{1}{2}}$; and

* By an affirmative power, is meant that whose Index is affirmative; and, by a negative power, that whose Index is negative.

and therefore, when One of these expressions is a Maximum, the Other will be a Maximum also. And (2^o.) since (*art.* 13.)

$$\left[\frac{a^2 c^2}{x^2} + a^2 + c^2 + x^2 \right]^{-1} \text{ is } = \frac{1}{\frac{a^2 c^2}{x^2} + a^2 + c^2 + x^2},$$

it is plain, that, the less the denominator of this fractional expression is, the greater will be the quotient; but the said denominator is evidently the least possible when $\frac{a^2 c^2}{x^2} + x^2$ is a Minimum.—Therefore, &c.

44. WHEN in the expression for the Fluxion of a Maximum or a Minimum there are two or more *fluxionary* letters, and each is contained in both affirmative and negative terms; then, the sum of the terms affected with either of them will be = 0.

Thus, if the Fluxion of a Minimum be $\frac{1}{2}ax - \dot{x}y + x\dot{y} - \frac{1}{2}by = 0$; then $\frac{1}{2}ax - \dot{x}y = 0$, and $x\dot{y} - \frac{1}{2}by = 0$.

For, let the variable rectangle CD, whose diagonal is AB, flow in the triangle EAF. Put FA = a , AE = b , AC or DB = x , and CB or AD = y . Then will the triangle FAB be = $\frac{1}{2}ax$, the triangle AEB be = $\frac{1}{2}by$, and the rectangle CD be = xy ; and therefore, the sum of the triangles FDB and BCE

will be $= \frac{1}{2}ax - xy + \frac{1}{2}by$; which, when the rectangle CD or xy is a Maximum, is evidently a Minimum. Now, it is plain, if x increases y must decrease, and therefore, (*art.* 22.) their Fluxions are negative to each other, and the Fluxion of the said Minimum is $\frac{1}{2}a\dot{x} - \dot{x}y + x\dot{y} - \frac{1}{2}b\dot{y} = 0$, and the Fluxion of the said Maximum is $\dot{x}y - x\dot{y} = 0$, or, the Fluxion of the triangle ADB or ACB is $\frac{1}{2}\dot{x}y - \frac{1}{2}x\dot{y} = 0$: but, the Fluxions of the triangles FAB and BAE are manifestly equal, since one increases as fast as the other decreases. Hence, therefore, (the Fluxion of the triangle ACB being $= 0$,) the Fluxion of the triangle FAB is $=$ the Fluxion of the triangle ECB; that is, $\frac{1}{2}a\dot{x} = \dot{x}y$, (*art.* 15. *demon.* 1^o.) consequently, $\frac{1}{2}a\dot{x} - \dot{x}y = 0$; and therefore $x\dot{y} - \frac{1}{2}b\dot{y} = 0$.

45. IN plane Figures, it is, in effect, the same thing to find the *greatest* Area that can be contained under a *given* Perimeter, as to find a *given* Area under the *least* Perimeter. — It may also be observed, that, to find the *greatest* Solid that can be contained under a *given* Surface, is the same as to find a *given* Solid under the *least* Surface.

46. GENERALLY, when a variable Quantity admits of a Maximum, it's Minimum is
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Nothing; and, when it admits of a Minimum, it's Maximum is Infinite.

EXAMPLE I.

47. To find the point C in the given right Fig. line AB, where the rectangle of the parts 26. AC and CB is a *Maximum*, or greater than any other rectangle $An \times nB$.

Put $AB = a$, and $AC = x$; then $CB = a - x$, and therefore the rectangle $AC \times CB = x \times a - x = ax - x^2 =$ a Maximum. Now, the Fluxion of a Maximum being $= 0$, the Fluxion of $ax - x^2$ must be $= 0$; that is, $a\dot{x} - 2x\dot{x} = 0$; which divided by $\dot{x}$, makes $a - 2x = 0$; therefore, $a = 2x$, and $x = \frac{1}{2}a$. Consequently, the rectangle of the parts AC and CB is a Maximum, or the greatest possible, when the said parts are equal.

Or, Put $AC = x$, and $CB = y$; then $AC \times CB = xy =$ a Maximum. Now, it is evident, that, if x increases y must decrease; and therefore the Fluxions of x and y are negative to each other; and consequently, when xy is a Maximum, the Fluxion of it will be $\dot{x}y - x\dot{y} = 0$: but, it is likewise evident, that, the Increment of x is equal to the Decrement of y ; or, that the Velocity with

with which x increases is equal to the Velocity with which y decreases; that is, $\dot{x} = \dot{y}$; therefore, by dividing by $\dot{x}$ or $\dot{y}$, or by striking both $\dot{x}$ and $\dot{y}$ out, in the above Fluxion of the Maximum xy , we shall have $y - x = 0$; and therefore $x = y$; as before.

Or, In the variable rectangle Ab , let the side Cb , which is equal and parallel to the side AD , always be equal to CB : then, while the rectangle is increasing by the motion of the decreasing side Cb moving from A towards B , it will be decreasing by the motion of the increasing side bD moving from F towards A . Now, since the Velocities of the points C and D , or the Fluxions of the sides AC and AD , are always equal, (as they evidently must be, because the sum of these sides is always the same;) therefore, as long as the side AC continues less than the side AD or Cb , the rectangle will be continually increasing; and after the side AC becomes equal to the side Cb , it will be continually decreasing: therefore, when the rectangle is a Maximum, or when it is neither increasing nor decreasing, the sides AC and Cb are equal to each other, or $AC = Cb = CB$; as before.

Or, Describe the semicircle AbB , and let fall the perpendicular bC . Now, by 35 E. 3.

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$AC \times CB = Cb^2$; and therefore $AC \times CB$ is a Maximum when bC is the radius of the semicircle, or when the point C bisects AB ; as before.

EXAMPLE II.

48. To find the point C in the given right line AB , where $AC^m \times CB^n$ is a *Maximum*.

Put $AB = a$, and $AC = x$; then $CB = a - x$, and $AC^m \times CB^n = x^m \times \overline{a-x}^n = \overbrace{x^{\frac{m}{n}} \times \overline{a-x}}^n =$ a Maximum; therefore, (*art.* 43.) $x^{\frac{m}{n}} \times \overline{a-x} =$ a Maximum; that is, (putting $\frac{m}{n} = e$,) $x^e \times \overline{a-x} = ax^e - x^{e+1} =$ a Maximum; the Fluxion of which is $= 0$; that is, $ae x^{e-1} \dot{x} - \overline{e+1} x^e \dot{x} = 0$. Now, by dividing this equation by $x^{e-1} \dot{x}$, we have $ae - \overline{e+1} x = 0$; therefore, $ae = \overline{e+1} x$, and $x = \frac{ae}{e+1}$; that is, by restitution, or writing $\frac{m}{n}$ for e , $x = \frac{m}{m+1} a$: whence the point C is determined.

Or, Put $AC = x$, and $CB = y$; then $x^m \times y^n =$ a Maximum; the Fluxion of which is $= 0$; that is, (because when x increases y de-

decreases, or the Fluxions of x^m and y^n are negative to each other,) $mx^{m-1}\dot{x} \times y^n - ny^{n-1}\dot{y} \times x^m = 0$: but, the Fluxions of x and y are evidently equal; therefore, throw both $\dot{x}$ and $\dot{y}$ out; then it will be $mx^{m-1} \times y^n - ny^{n-1} \times x^m = 0$. Now, by dividing this equation by $x^{m-1}y^{n-1}$, we have $my - nx = 0$; therefore, $nx = my$, and $m : n :: x : y$. So that, the segments AC and CB are in direct proportion to the Indices of their Powers; and $x = \frac{m}{n}y = \frac{m}{n} \times a - x = \frac{ma - mx}{n}$; therefore, $nx = ma - mx$, and $x = \frac{m}{m+n}a$; as before.

EXAMPLE III.

49. To find the *greatest* Cone that can be inscribed in a given Sphere.

Fig. Put AD, the diameter of the Sphere, $= a$;
 27. .78539 &c. (the area of a circle whose diameter is 1,) $= c$; and AC, the altitude of the cone, $= x$: then $CD = a - x$. Now, by 35 E. 3. $AC \times CD = CB^2$; that is, $x \times a - x = ax - x^2 = CB^2$; therefore, (because the square of the diameter is 4 times the square of the radius,) by 2 E. 12. $4acx - 4cx^2 =$ the area of the cone's base; which, by

10 E. 12, drawn into $\frac{1}{3}x$, is $\frac{4}{3}acx^2 - \frac{4}{3}cx^3$ = the cone's solidity, a Maximum: therefore, by art. 43. $ax^2 - x^3$ = a Maximum; the Fluxion of which is = 0; that is, $2axx - 3x^2\dot{x} = 0$; which divided by xx , gives $2a - 3x = 0$; therefore, $3x = 2a$, and $x = \frac{2}{3}a$. So that, the cone will be a Maximum, or the greatest possible, when it's altitude is = two-thirds of the sphere's diameter.

EXAMPLE IV.

50. To find the internal dimensions of a cylindrical Cup, whose capacity is given = a , when the said Cup is made with the *least* possible quantity of Silver of a given thickness.

Put the diameter = x , and .78539 &c. (the area of a circle whose diameter is 1,) = c ; then, by 2 E. 12. cx^2 = the area of the bottom, and therefore $\frac{a}{cx^2}$ = the altitude: but, $4cx$ = the circumference of the bottom, and therefore $4cx \times \frac{a}{cx^2} = \frac{4a}{x}$ = the inside curve superficies. Hence $cx^2 + \frac{4a}{x}$ = the whole inside superficies; which, because the

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quantity of silver is the Least possible, is a Minimum; and therefore it's Fluxion is $=0$;

that is, $2cx\dot{x} - \frac{4a\dot{x}}{x^2} = 0$; which multiplied by x^2 , is $2cx^3\dot{x} - 4a\dot{x} = 0$; and this divided by $2\dot{x}$, is $cx^3 - 2a = 0$; therefore, $cx^3 = 2a$, and $x = \sqrt[3]{\frac{2a}{c}}$ = the diameter; which substi-

tuted for x , makes the above $\frac{a}{cx^2} = \frac{a}{c \times \sqrt[3]{\frac{2a}{c}}}$

$$= \frac{a \times \sqrt[3]{\frac{2a}{c}}}{c \times \frac{2a}{c}} = \frac{1}{2} \times \sqrt[3]{\frac{2a}{c}} = \text{the altitude. So}$$

that the diameter is to the altitude as 2 to 1.

Or, Put the diameter $= x$, altitude $= y$, and $.78539 \&c. = c$; then, the whole inside superficies will be $= cx^2 + 4cxy =$ a Minimum; therefore (*art. 43.*) $x^2 + 4xy =$ a Minimum; the Fluxion of which is $= 0$; that is, (supposing x to increase and y to decrease, or the Fluxions of x and y to be negative to each other, as they evidently must be,) $2x\dot{x} + 4\dot{x}y - 4x\dot{y} = 0$. But, $cx^2y = a$, and consequently $x^2y = \frac{a}{c}$; the Fluxion of which equation,

tion, (because the Fluxion of $\frac{a}{c}$ is $=0$, and the Fluxion of y is negative,) is $2x\dot{x}y - x^2\dot{y} = 0$; therefore $x^2\dot{y} = 2x\dot{x}y$; which equation divided by x^2 , makes $\dot{y} = \frac{2\dot{x}y}{x}$. Hence, this value of $\dot{y}$ being substituted for it in the above Fluxion of the Minimum, we have $2x\dot{x} + 4\dot{x}y - 8\dot{x}y = 0$; that is, $2x\dot{x} - 4\dot{x}y = 0$; which equation divided by $2\dot{x}$, gives $x - 2y = 0$; consequently $x = 2y$; as before.

EXAMPLE V.

51. To find the internal dimensions of a Cistern in the form of a rectangular solid, (that is, whose bottom and all four sides are rectangular,) when it's capacity is $=a$, and it is made with the *least* possible quantity of Lead of a given thickness.

Put the inside length AC or DB $=x$, *Fig.* breadth AD or CB $=y$, and depth BF or CE $=z$.

$=z$; then, $xyz = a$, and therefore, $x = \frac{a}{yz}$.

Now, the inside superficies of the bottom and four sides is $= AC \times CB + 2AC \times CE + 2CB \times BF = xy + 2xz + 2yz =$ (by substituting $\frac{a}{yz}$ for x ,) $\frac{a}{z} + \frac{2a}{y} + 2yz$; which, because

the cistern is made with the Least possible quantity of lead, is a Minimum; and therefore it's Fluxion is $=0$; that is, $-\frac{az}{z^2}$

$-\frac{2ay}{y^2} + 2yz + zy\dot{z} = 0$. Now, (*art.* 44.) the sum of the terms affected with $\dot{y}$ is $=0$, and the sum of the terms affected with $\dot{z}$ is $=0$; that is, $-\frac{2ay}{y^2} + 2yz = 0$, and $-\frac{az}{z^2} + 2yz = 0$; the former of which equations multiplied by y^2 , gives $-2ay + 2y^2yz = 0$, and this divided by $2y$, is $-a + y^2z = 0$, $\therefore y^2z = a = xyz$, therefore, $y = x$: and the latter of the said equations multiplied by z^2 , gives $-az + 2yz^2\dot{z} = 0$; which divided by $\dot{z}$, is $-a + 2yz^2 = 0$, $\therefore 2yz^2 = a = xyz$, therefore, $2z = x$. Hence, $x = y = 2z$; that is, the length and breadth will be equal, and each equal to twice the depth.

Or, The length, breadth, and depth, being x , y , and z , respectively; and the inside superficies $= xy + 2xz + 2yz =$ a Minimum, as before; if we suppose x and y to increase, then z must necessarily decrease; that is, the Fluxions of x and y being affirmative, the Fluxion of z will be negative; and therefore, the Fluxion of the Minimum will be

xy

$\dot{x}y + x\dot{y} + 2\dot{x}z - 2x\dot{z} + 2\dot{y}z - 2y\dot{z} = 0$. But, $xyz = a$, the Fluxion of which equation, (the Fluxion of a being $= 0$, and the Fluxion of z being negative,) is $\dot{x}yz + x\dot{y}z - xy\dot{z} = 0$; $\therefore \dot{z} = \frac{\dot{x}yz + x\dot{y}z}{xy} = \frac{\dot{x}z}{x} + \frac{\dot{y}z}{y}$, which substituted for $\dot{z}$, in the Fluxion of the Minimum, makes it $\dot{x}y + x\dot{y} - \frac{2x\dot{y}z}{y} - \frac{2\dot{x}yz}{x} = 0$; therefore (art. 44.) $\dot{x}y - \frac{2\dot{x}yz}{x} = 0$, and $x\dot{y} - \frac{2x\dot{y}z}{y} = 0$. Now, the first of these two equations multiplied by $\frac{x}{\dot{x}y}$, is $x - 2z = 0$, $\therefore x = 2z$: and the latter multiplied by $\frac{y}{x\dot{y}}$, is $y - 2z = 0$, $\therefore y = 2z$. So that, $x = y = 2z$; as before.

Scholium.

In this last method of Solution, either of the quantities x , y , or z , might have been supposed to decrease while the others increase; or, but one of them to increase while the others decrease: for, in either case, the conclusions would come out exactly the same.

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EXAMPLE VI.

Fig. 29. 52. A Gentleman wants to ride from the City C to the City D, the Cities being a miles apart. Now, from C to A, which is b miles, or from C to any place B in the road DA, which is perpendicular to the road AC, he can ride after the rate of c miles an hour; but, from A to D he can ride faster, or after the rate of d miles an hour. To find B, the place to which he must directly ride, in order to perform his Journey in the *least* possible Time.

By 47 E. 1. $AD = \sqrt{DC^2 - CA^2} = \sqrt{a^2 - b^2}$,
which put $= e$; and let $AB = x$; then BD
 $= e - x$; and therefore, $\frac{e-x}{d}$ = the num-

ber of hours the Gentleman will be riding from B to D. Again, by 47 E. 1. $CB =$

$\sqrt{BA^2 + AC^2} = \sqrt{x^2 + b^2}$; and therefore,
 $\frac{\sqrt{x^2 + b^2}}{c}$ = the number of hours he will be

riding from C to B. Hence we have $\frac{\sqrt{x^2 + b^2}}{c}$

$+$ $\frac{e-x}{d}$ = the number of hours he will be

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performing his journey, which must be a Minimum: therefore (*art.* 43.) $d \times \overline{x^2 + b^2}^{\frac{1}{2}} - cx = \text{a Minimum}$; the Fluxion of which is $= 0$; that is, $\frac{1}{2} d \times \overline{x^2 + b^2}^{-\frac{1}{2}} \times 2x\dot{x} - c\dot{x} = 0$; which multiplied by $\overline{x^2 + b^2}^{\frac{1}{2}}$ gives $dx\dot{x} - c\dot{x} \times \overline{x^2 + b^2}^{\frac{1}{2}} = 0$; and this transposed and divided by $\dot{x}$, is $dx = c \times \overline{x^2 + b^2}^{\frac{1}{2}}$; the square of which is $d^2 x^2 = c^2 x^2 + b^2 c^2$; therefore $x^2 = \frac{b^2 c^2}{d^2 - c^2}$, and $x = \frac{bc}{\overline{d^2 - c^2}^{\frac{1}{2}}} = AB$.

Or, Let Cb be supposed indefinitely near to CB , and the little circular arch Bn be described with the radius CB ; that is, let Bb express the nascent or indefinitely small Increment of AB , and nb that of CB . Now, it is evident, if B be the place to which he must directly ride, that, the Increments Bb and nb must be to each other as d to c , since then only can they be passed over in the same time: but, the little triangle Bnb is (or may be taken as) similar to the right angled triangle CAB ; (for, the arch Bn ,
being

being indefinitely small, may be considered as a little right line perpendicular to Cb ; so likewise, the angle BCb being indefinitely little, the angle CbA or nbB may be considered as equal to the angle CBA ; and therefore the angle bBn as equal to the angle BCA ; consequently, by 4 E. 6. $d : c :: CB : BA$; but, when the hypotenuse and perpendicular are d and c , the base, by 47 E. 1. will be $\sqrt{d^2 - c^2}^{\frac{1}{2}}$. Hence, therefore, $\sqrt{d^2 - c^2}^{\frac{1}{2}} : (CA) b :: c : \frac{bc}{\sqrt{d^2 - c^2}^{\frac{1}{2}}} = AB$; as before.

Construction.

Through C draw mv parallel to AD ; making $Cv = d + c$, and $Cm = d - c$; describe the semicircle mkv ; on the intersecting point k erect the perpendicular $kr = c$; and through r draw the right line CB : then will B be the point required. For, by 35 E. 3. $\sqrt{Cv \times Cm}^{\frac{1}{2}} = Ck$, that is, $\sqrt{d^2 - c^2}^{\frac{1}{2}} = Ck$; and, by 4 E. 6. $Ck : kr :: CA : AB$, that is, $\sqrt{d^2 - c^2}^{\frac{1}{2}} : c :: b : \frac{bc}{\sqrt{d^2 - c^2}^{\frac{1}{2}}} = AB$.

EXAMPLE VII.

53. Let the triangle BAD have one angle A in the right line CE: To find a *Maximum* of the Sum of the perpendiculars BC and DE let fall from the other two angles on the right line aforesaid. Fig.
30.

Note, $AB = a$, $AD = b$, and the $\angle BAD = 90^\circ$.

Put $AC = x$; then, by 47 E. 1. $CB = \sqrt{BA^2 - AC^2}^{\frac{1}{2}} = \sqrt{a^2 - x^2}^{\frac{1}{2}}$. Now, because the $\angle BAD$ is right, the $\angle DAE$ is the complement of the $\angle BAC$, and is therefore $= \angle ABC$; consequently the triangles BAC and ADE are similar; and therefore, by 4 E. 6. $BA : AC :: AD : DE$, that is, $a : x :: b : \frac{bx}{a} = DE$. Hence we have $BC + DE = \sqrt{a^2 - x^2}^{\frac{1}{2}} + \frac{bx}{a} = \text{a Maximum}$; the Fluxion

of which is $= 0$; that is, $\frac{1}{2} \times \sqrt{a^2 - x^2}^{-\frac{1}{2}} \times -2x\dot{x} + \frac{b\dot{x}}{a} = 0$, that is, $-\frac{x\dot{x}}{\sqrt{a^2 - x^2}^{\frac{1}{2}}} + \frac{b\dot{x}}{a} = 0$; which multiplied by $\sqrt{a^2 - x^2}^{\frac{1}{2}} \times a$, gives $-ax\dot{x} + \sqrt{a^2 - x^2}^{\frac{1}{2}} b\dot{x} = 0$; therefore,

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by

74 *An INTRODUCTION to the*

by transposition and dividing by x , we have

$$ax = \sqrt{a^2 - x^2}^{\frac{1}{2}} b; \text{ by involution, } a^2 x^2 = a^2 b^2 - b^2 x^2; \text{ consequently, } a^2 x^2 + b^2 x^2 = a^2 b^2, \\ \text{and } x^2 = \frac{a^2 b^2}{a^2 + b^2}; \text{ therefore, } x = \frac{ab}{\sqrt{a^2 + b^2}^{\frac{1}{2}}};$$

which substituted for x , makes the Maximum $BC + DE (= \sqrt{a^2 - x^2}^{\frac{1}{2}} + \frac{bx}{a}) = \sqrt{a^2 + b^2}^{\frac{1}{2}}$

$= \sqrt{BA^2 + AD^2}^{\frac{1}{2}}$, that is, by 47 E. I. $BC + DE =$ the hypotenuse BD .

Or, Produce BA to b , making $bA = AB$; draw the right line bD ; and perpendicular to CE draw bc : then will the triangles Abc and ABC be equal and similar, for bc is equal and parallel to BC . Now, it is evident, that, the line bD , in every situation, will be greater than the sum of the perpendiculars bc and DE , excepting when it is perpendicular to the line CE ; and that then, the said perpendiculars will coincide with, and their sum be equal to it. Therefore, the Maximum of the Sum of the perpendiculars bc and DE , or of BC and DE , is equal to the side bD of the triangle $A b D$; which, when the $\angle bAD$ or BAD is right, is equal to BD ; as before.

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EXAMPLE VIII.

54: To find the *greatest* right angled Tri-*Fig.* angle ACB that can be inscribed in the 31. given Quadrant AEF; the right angle being formed by the sine and cosine BC and CA.

Put the radius or hypotenuse $AB=a$, and $AC=x$; then, by 47 E. 1. $CB = \sqrt{a^2 - x^2}^{\frac{1}{2}}$; therefore, the area of the triangle $= \frac{1}{2} x \times \sqrt{a^2 - x^2}^{\frac{1}{2}} = \frac{1}{4} a^2 x^2 - \frac{1}{4} x^4^{\frac{1}{2}} =$ a Maximum: consequently, (*art.* 43.) $a^2 x^2 - x^4 =$ a Maximum; the Fluxion of which is $= 0$; that is, $2a^2 x \dot{x} - 4x^3 \dot{x} = 0$; which divided by $2x \dot{x}$, makes $a^2 - 2x^2 = 0$; therefore $x^2 = \frac{1}{2} a^2$; consequently $AC=CB$, and the point B bisects the quadrantal arch EF.

Or, Put $AB=a$, $AC=x$, $CB=y$, $EB=z$; and suppose the point b indefinitely near to B, bc parallel to BC, and Bn equal and parallel to Cc ; that is, let $Bn=x'$, and $Bb=z'$. Now, if we consider the Increment Bb as a little right line perpendicular to the radius AB, the angles nBb and CBA will be equal, and consequently the indefinitely little right angled triangle Bnb will be similar to the right angled triangle BCA; therefore, by

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4 E. 6.

76 *An INTRODUCTION to the*

4 E. 6. $AB : BC :: bB : Bb$, that is, $a : y ::$

$z' : x'$, or (*art. 7.*) $a : y :: z : x$, $\therefore z = \frac{ax}{y}$.

But, when the triangle ACB is a Maximum, it is plain, that, $AB \times \frac{1}{2} z$ is $= BC \times x$; that

is, $\frac{1}{2} az = yx$, $\therefore z = \frac{2yx}{a}$ (by the above,)

$\frac{ax}{y}$; whence, $2y^2x = a^2x$, and $2y^2 = a^2 = x^2$

$+ y^2$. Therefore, $x = y$, or $AC = CB$; as before.

Or, Suppose CD to be always perpendicular to AB. Now, it is evident, the Triangle will be the *greatest* when CD is a Maximum, that is, when it bisects AB, or is the radius of the circumscribing semicircle ACB; and then, it is plain, $AC = CB$; as before.

EXAMPLE IX.

Fig. 55. In the right line AE, which is perpendicular to the indefinite right line AZ,
32. given $AC = a$, and $CE = b$: To find the point B, where the Angle CBE is a Maximum.

Put $AE = a + b = c$, and $AB = x$; then, by

47 E. 1. $CB = \sqrt{a^2 + x^2}^{\frac{1}{2}}$, and $EB = \sqrt{c^2 + x^2}^{\frac{1}{2}}$.

Now,

Now, by Trigonometry, $CB : \text{radius} :: AB : s. \angle ACB$, that is, (putting radius = 1,)

$$\sqrt{a^2 + x^2}^{\frac{1}{2}} : 1 :: x : \frac{x}{\sqrt{a^2 + x^2}^{\frac{1}{2}}} = s. \angle ACB \text{ or}$$

ECB ; and $EB : s. \angle ECB :: EC : s. \angle CBE$, that

$$\text{is, } \sqrt{c^2 + x^2}^{\frac{1}{2}} : \frac{x}{\sqrt{a^2 + x^2}^{\frac{1}{2}}} :: b : \frac{bx}{\sqrt{a^2 + x^2}^{\frac{1}{2}} \times \sqrt{c^2 + x^2}^{\frac{1}{2}}} \\ = s. \angle CBE = \text{a Maximum; therefore (art. 43.)}$$

$$\frac{x^2}{a^2 + x^2 \times c^2 + x^2} = \frac{1}{a^2 c^2 x^{-2} + a^2 + c^2 + x^2} =$$

a Maximum; and $a^2 c^2 x^{-2} + x^2 =$ a Minimum; the Fluxion of which is $= 0$; that is, $-2a^2 c^2 x^{-3} \dot{x} + 2x \dot{x} = 0$; and this equation divided by $2x \dot{x}$, makes $-a^2 c^2 x^{-4} + 1 = 0$;

$$\text{therefore } 1 = a^2 c^2 x^{-4} = \frac{a^2 c^2}{x^4}, \text{ and } x^4 =$$

$a^2 c^2$, and $x = \sqrt{ac}^{\frac{1}{2}}$. So that AB is a geometrical mean between the distances AC and AE ; and therefore, by 6 E. 6. the triangles CAB and BAE are similar.

Or *, (in any position of the line AZ ,) — *Fig.*
 It is evident, that, as long as the angle BCE 32.
 decreases faster than the angle BEC in- 33.
 creases, the angle CBE will be increasing; 34.
 and that, when the angle BEC increases
 faster

* Invented Anno 1760.

faster than the angle BCE decreases, the angle CBE will be decreasing: therefore, when the said angle CBE is a Maximum, or when it neither increases nor decreases, the Fluxions of the angles at C and E will be equal; or, the Increment of the angle BEC will be equal to the Decrement of the angle BCE; that is, if we suppose the point *b* indefinitely near to B, the $\angle BEb = \angle BCb$: therefore, if with EB and CB, as radii, the arches *Bm* and *Bn* be described; then, because the circumferences of circles are as their radii, $Bm : Bn :: EB : CB$. Describe the circle BGA, whose diameter is AB; and in *fig.* 34. produce EB to G, and BC to I. Draw the right lines GA, AI, and IG. Then, in *fig.* 32 and 33. (by 21 E. 3.) $\angle GAI = \angle GBI$ or EBC; and in *fig.* 34. (by 22 E. 3.) $\angle GAI + \angle GBI = 180^\circ. = \angle GBI + \angle EBI$; therefore in this *fig.* also, $\angle GAI = \angle EBI$ or EBC. Now, if we suppose the arches *Bm* and *Bn* to be little right lines perpendicular to *Em* and *Cn* respectively, then may the triangles *bBm* and BAG be considered as similar, as may likewise the triangles *bBn* and BAI; (for, by 31 E. 3. the angles BGA and BIA are right; and, because the points *b* and B are presumed to be indefinitely near

to

to each other, therefore the $\angle EbA$ is indefinitely near to equality with the $\angle EBA$, and the $\angle CbA$ with the $\angle CBA$; &c.) wherefore, by 4 E. 6. $bB : BA :: Bm : AG$, and $bB : BA :: Bn : AI$; $\therefore Bm : Bn :: AG : AI$. Hence $EB : CB :: AG : AI$; and therefore, by 6 E. 6. the triangles EBC and GAI are alike, and the $\angle AGI = \angle BEC$: but, by 21 E. 3. the $\angle AGI = \angle ABI$ or ABC ; therefore the $\angle BEC = \angle ABC$, and consequently the triangles CAB and BAE are similar; wherefore, by 4 E. 6. $CA : AB :: BA : AE$; $\therefore AB = \sqrt{AC \times AE}^{\frac{1}{2}}$.

Construction.

Produce EA to K , making $AK = AC$; describe the semicircle KE ; and, in *fig.* 33 and 34. draw AQ perpendicular to the diameter EK , and make AB equal to AQ : then will B be the point required. For, by 35 E. 3. $AQ = \sqrt{KA \times AE}^{\frac{1}{2}}$, that is, $AB = \sqrt{AC \times AE}^{\frac{1}{2}}$.

EXAMPLE X.

56. Given the point C in the radius EA of *Fig.*
the circle AB &c. To find the point B ,
at _{35.}

at which, if a tangent TB and right line CB be drawn to it, the Angle CBT is a *Minimum*.

Draw the radius EB. Now, because by 18 E. 3. the tangent to a circle is always perpendicular to the radius, it is evident, that, the angle CBT will be the least possible when the angle CBE is the greatest: It is likewise evident, that, when the said angle CBE is the greatest possible, the Fluxions of the angles BCE and BEC will be equal; or, supposing the point *b* indefinitely near to B, $\angle BCB = \angle BEb$; and therefore, if with CB, as a radius, we describe the little circular arch *Bn*; then, $CB : Bn :: EB : Bb$, or, $CB : BE :: nB : Bb$; and consequently, by 6 E. 6. (because $\angle CBn = \angle EBb$, and therefore $\angle CBE = \angle nBb$,) the triangles CBE and *nBb* are similar, and $\angle ECB = \angle bnB =$ a right angle. Hence, when the angle CBE is a Maximum, or the angle CBT is a Minimum, the right line CB will be perpendicular to the radius EA.

Or, Because by Trigonometry $EB : \text{fine} \angle BCE :: EC : \text{fine} \angle CBE$; therefore, the angle CBE will be the greatest when the angle ECB is a right one; but, when the
angle

angle CBE is the greatest, the angle CBT is the least possible: therefore, when the said angle CBT is a Minimum, the right line CB will be perpendicular to the radius EA; as before.

EXAMPLE XI*.

57. Given the point C in the radius OA of the semicircle ABE: To find the point B in the said semicircle where the Sum of the right lines CB and EB is a *Maximum*. Fig. 36.

Suppose the point *b* indefinitely near to B, and the little circular arches B*n* and B*m*, described with CB and EB as radii, to be little right lines perpendicular to the said radii respectively. Now, the angles CB*n*, OB*b*, and EB*m*, being right; therefore $\angle CBO = \angle nBb$, and $\angle OBE = \angle mBb$: but, when $CB + EB$ is a Maximum, it is evident, that, the Increment *nb* must be = the Decrement *bm*; and therefore the triangles *bnB* and *bmB* are equal and similar. Hence, $\angle nBb$ being = $\angle mBb$, therefore $\angle CBO = \angle OBE = \angle OEB$; and the triangles OCB and BCE are similar; and consequently, by 4 E. 6. $OC : CB :: BC : CE$; $\therefore CB = \sqrt{CO \times CE}$.

Con-

* Invented Anno 1761.

Construction.

Make $CK=CO$; describe the semicircle KQE , and draw the right line CQ perpendicular to the diameter EK ; also, make $CB=CQ$: then will B be the point required.

For, by 35 E. 3. $CQ=\sqrt{EC \times CK}^{\frac{1}{2}}$, that is, $CB=\sqrt{EC \times CO}^{\frac{1}{2}}$.

Corollary.

In order to make the Maximum take place, the given distance OC must be greater than $\frac{1}{3}$ of the radius OA .

Note.

To find an Algebraical expression for the sum of the right lines CB and EB when the said sum is a Maximum. Put the radius EO or $OA=a$, and $OC=b$: then, $CB=$

$$\sqrt{CO \times CE}^{\frac{1}{2}} = \sqrt{ab + b^2}^{\frac{1}{2}}, \text{ and } CB : BO :: CE : EB, \text{ or } EB = \frac{BO \times CE}{CB} = \frac{a^2 + ab}{\sqrt{ab + b^2}^{\frac{1}{2}}}; \text{ therefore}$$

$$CB + EB = \sqrt{ab + b^2}^{\frac{1}{2}} + \frac{a^2 + ab}{\sqrt{ab + b^2}^{\frac{1}{2}}} = \frac{\overline{a+b}^2}{b^{\frac{1}{2}} \cdot \overline{a+b}^{\frac{1}{2}}} \\ = \frac{\overline{a+b}^3}{b}^{\frac{1}{2}}.$$

Ex-

EXAMPLE XII*.

58. Let the given right line CG in it's first *Fig.* situation coincide with the right line AE; 37. and let the end G move along the right line ED in such a manner that the other end C may pass with an uniform motion from A to E; and, at the same time, suppose a point B to move with the same uniform motion from C along the line CG; then, by the motion of this point, will the Curve ABD be described: To find the point C in the line AE where CF is a *Maximum*; BF being always parallel to DE, and BH parallel to AE.

Put $AE=ED=CG=a$, $AF=x$, and $AC=CB=z$; then, $CF=x-z$, $BG=a-z$, and $FE=BH=a-x$. Now, by 4 E. 6. $GB: BH :: BC: CF$, that is, $a-z: a-x :: z: x-z$; $\therefore ax-az-xz+z^2=az-xz$, or $ax-2az+z^2=0$; the Fluxion of which equation is $a\dot{x}-2a\dot{z}+2z\dot{z}=0$: but, when CF or $x-z$ is a Maximum, it is evident, that, $\dot{x}=\dot{z}$; therefore, by dividing by $\dot{x}$ or $\dot{z}$, we have $a-2a+2z=0$, or $2z=a$; and therefore $z=\frac{1}{2}a$, or, the point C bisects the line AE when CF is a Maximum.

Corollary.

* Invented Anno 1756.

Corollary.

Since $CB = BG$; therefore $CF = BH = FE = \frac{1}{2}CB$; and consequently, when the angle AED is not an Obtuse but a Right one, the angle BCF will, when CF is a Maximum, be $= 60^\circ$.

EXAMPLE XIII.

- Fig. 38. 59. To find the point of Retrogression B in the *Contracted Semicycloid* ABD; whose generating semicircle AGF is greater than it's base FD*.

Put the generating semicircle $AGF = a$,
base $FD = b$, radius $OG = c$, $CG = s$, $OC = x$,
or-

* This Curve may be thus generated.—Let the semicircle af roll along upon the right line fd equal to it and perpendicular to it's diameter fa : then will the curve ABD described by any point A taken *without* the said semicircle and in the radius Oa produced, be a *Contracted Semicycloid*. For, describe the concentric semicircle AGF; in any position of which, as BRK, to the generating point B draw the ordinate CB parallel to the base FD, which said base must evidently be equal and parallel to fd : through the center o draw MR parallel to the diameter AF; and with the radius oB describe the arch BM: then will the arches AG, MB, and RK, be equal, and $CB = CG$; and therefore $GB = Ct = fr =$ (by the generation) arch rk ; but, semicircle AF : semicircle $af ::$ arch RK : arch rk ; therefore, semicircle AF : fd or base FD :: arch AG : GB; which is the property of the *Cycloid*. (See art. 35.)

This Curve may, with propriety, be called an *exterior Cycloid*.

ordinate $CB=y$, and arch $AG=z$; and let the point g be supposed indefinitely near to G , and ng parallel to CF ; that is, let $nG=s'$, and $Gg=z'$. Now, if the Increment Gg be supposed a little right line perpendicular to the radius OG , the right angled triangles OCG and Gng will be similar; and therefore, by 4 E. 6. $CO : OG :: nG : Gg$, that is, $x : c :: s' : z'$, or (art. 7.) $x : c :: s : \dot{z} = \frac{cs}{x}$. By the known property of

the curve, $a : b :: z : GB = \frac{bz}{a}$; therefore

$CG + GB = s + \frac{bz}{a} = y$, which, at the point of Retrogression, must evidently be a Maximum; it's Fluxion therefore is $=0$, that is, because the Fluxions of s and z are negative to each other, $-\dot{s} + \frac{b\dot{z}}{a} = 0$; from

which equation we have $\dot{z} = \frac{a\dot{s}}{b}$. Hence $\frac{cs}{x} = \frac{a\dot{s}}{b}$; and therefore $ax = bc$, and $x = \frac{bc}{a} = OC$.

Or, * Put the generating semicircle $AGF = a$, base $FD = b$, radius $OG = c$, and $OC = x$; and let gb be supposed indefinitely near

* Invented Anno 1760.

near and parallel to GB. Now, it is evident, that, at the point of Retrogression B, the tangent must be perpendicular to the ordinate; that is, at the said point, the Increment Bb is perpendicular to bv ; and therefore, because the $\angle tBb = \angle oBv$, the right angled triangles Bbv and Bto or GCO are similar; and consequently, by 4 E. 6. $Bv : vb :: GO : OC$. But, by the nature or generation of the curve, $a : b :: Gg$ or $Bv : vb$. Hence therefore, $a : b :: GO : OC$, that is, $a : b :: c : x$; $\therefore x = \frac{bc}{a} = OC$; as before.

Corollaries.

1. At any point B in the curve, if to the corresponding point G in the circle, we draw the right line TG perpendicular to the radius OG, making $TG : GB :: a : b$, that is, if TG be made equal to the arch AG; or if Bx be made equal to the radius Oa, and xz be drawn parallel to the tangent GT and equal to the radius OA; then will the right line drawn from T or z to B, be a Tangent to the curve at the point B. For then the triangles TGB or zxB and Bvb will be similar; and therefore, &c.

2. If the right line aQ be drawn perpendicular to the radius OA ; then, when the point Q arrives at the base FD , the point A will be in the point of Retrogression B . For, since at the said point of Retrogression, $x = \frac{bc}{a}$; by analogy, $a : b :: c : x$; that is, semicircle BRK : semicircle $prk :: oR : ot$; therefore, the semicircles being as their radii oR and or , the point t coincides with r , that is, the ordinate CB coincides with the right line fd ; and the triangles Rop and Bot are equal and similar. Consequently, Rp is perpendicular to po ; and therefore, &c.—Hence the following

Construction.

Make $DR = \text{arch } ae$; draw Rt equal and parallel to Ff , and tB equal and parallel to aQ : then will B be the point of Retrogression required.

C H A P. V.

Of finding the Points of Inflection in Curves.

60. WHEN a Curve from being Concave becomes Convex towards it's axis; or, from
being

being Convex becomes Concave; then, that Point in it where the Change is made, or that which separates the Convex from the Concave part, is called the Point of *Inflexion*. So that, if to the Point of Inflexion a Tangent be drawn, it will cut the Curve.

Thus, in *fig. 39.* if AB be concave and
Fig. 39. BY convex, or, in *fig. 40.* if AB be convex
 39. and BY concave towards the axis AX; then
 40. B is the point of Inflexion: where the Tangent TBG cuts the curve.

61. Now, it is evident, that, in any Curve, in order to determine whether the Absciss or Ordinate flows with an accelerated or retarded motion, or, to find the value of it's *Second Fluxion*, it is necessary that one of them be made to increase or decrease with a given or uniform motion, with which the swiftness of the increase or decrease of the other may be always compared.

Suppose the Absciss therefore, it being the most natural, always to flow equably, or equal parts of it to be described in equal times; then, because the Direction of the curve from A to B (*fig. 39.*) or from B to Y (*fig. 40.*) continually approaches nearer to a Parallelism with the axis, it is evident, the ordinate between these points must flow
 with

with a motion continually Retarded: and, because the Direction of the curve from B to Y (*fig. 39.*) or from A to B (*fig. 40.*) approaches continually nearer to a Perpendicular to the axis; therefore, between these points, the ordinate must flow, or increase, with a motion continually Accelerated. Consequently, at the point of Inflection, the ordinate will flow with neither an Accelerated nor Retarded but with an Uniform motion: therefore, at the point of Inflection B, the *Second Fluxions* of the absciss and ordinate will be $=0$.

Or, Let Bn be a given right line always parallel to the base, Bm a Tangent to the curve, and nm a right line parallel to the ordinate. Then, it is plain, that, before the ordinate arrives at the point of Inflection, the right line nm , in *fig. 39.* will be continually Decreasing, and afterwards continually Increasing; or, in *fig. 40.* will be continually Increasing, and afterwards continually Decreasing: therefore, at the point of Inflection, it will be neither Increasing nor Decreasing; but will, in *fig. 39.* be a Minimum, or, in *fig. 40.* a Maximum; and consequently, in either case, it's Fluxion will be $=0$. But, by *art. 24.* the right lines Bn ,

N

 nm ,

nm , and Bm , are as the Fluxions of the absciss, ordinate, and curve, respectively. Hence therefore, the *Second Fluxions* of the absciss and ordinate, at the point of Inflection, are $=0$; as before.

62. Now, since, in general, the ordinate CB and the measure of it's Fluxion, nm , flow together; or, since the Fluxion of the ordinate CB is always as the right line nm , and the said right line is a Variable or Flowing quantity*; therefore, to find the *Second Fluxion* of a *Fluent*, or the *Fluxion* of an Expression containing the *First Fluxions* of any variable quantities; every *First Fluxion*, not supposed Invariable, must be considered as a distinct Variable quantity, and the Expression

* If the right line nm be invariable; then, the curve will degenerate into a right line, and the ordinate will flow uniformly, or the Fluxion of it be always the same.

If the right line nm be variable; then, the velocity with which the ordinate flows will likewise be variable: And, the velocity with which the line nm increases or decreases will always be as the increase or decrease of the velocity with which the ordinate flows; that is, the Fluxion of the line nm will be as the Second Fluxion of the ordinate CB.

If the right line nm does not uniformly increase or decrease; then, the velocity with which the ordinate flows will not uniformly increase or decrease: And, the increase or decrease of the velocity with which the line nm increases or decreases will always be as the increase or decrease of the acceleration or retardation of the velocity with which the ordinate flows; that is, the Second Fluxion of the line nm will be as the Third Fluxion of the ordinate CB.

sion be put into Fluxions by the *Rules* laid down in *Chap. 2*.

Thus, the Fluxion of $2y\dot{y}$ is $2y\ddot{y} + 2y\dot{y}$, that is, $2y^2 + 2y\ddot{y}$; or, if y be invariable, it is $2y^2$.

The Fluxion of $\frac{\dot{x}y}{\dot{y}}$ is $\frac{\ddot{x}y\dot{y} + \dot{x}\dot{y}^2 - \ddot{y}xy}{\dot{y}^2}$; or, since

either $\dot{x}$ or $\dot{y}$ may be supposed invariable, (that is, either x or y to flow with an uniform motion,) if we make $\dot{x}$ invariable, it will be $\frac{\dot{x}\dot{y}^2 - \ddot{y}xy}{\dot{y}^2}$. Also, the Fluxion of

$$\frac{x\dot{x}}{a^2 + y^2} \text{ is } \frac{\dot{x}^2 + x\ddot{x}a^2 + y^2}{a^2 + y^2} \frac{\dot{x}}{2} - \frac{y\dot{y}}{a^2 + y^2} x\ddot{x}.$$

THE Third and Fourth Fluxions, &c. are found in the same manner, due regard being had to such Fluxions as are supposed Invariable.

63. HENCE, to find the point of Inflection B; put the Equation of the curve (where the absciss AC is $=x$ or $a \pm x$, and the ordinate CB $=y$.) into Fluxions; from which, or from other properties of the curve, find the value of $\dot{x}$ or $\dot{y}$; and put this $\dot{x}$ or $\dot{y}$ and it's value into Fluxions, making both $\ddot{x}$ and $\ddot{y} = 0$: then, by expunging the rest of the Fluxional quantities,

ties, you may have x or y , at the point of Inflection sought, determined.

64. BUT, the point of Inflection may be found without the help of Second Fluxions. For, if TB be a Tangent to it, it is evident, that, AT, the Difference between the subtangent and absciss, will be a *Maximum*. And therefore, to find the point of Inflection, we need only to find a definitive expression for the Subtangent, by *Chap. 3.* and the Difference between it and the absciss; and then make the Fluxion of this Difference $= 0$.—And sometimes it may be determined in a *New* and different manner; as will be shewn in the following Example.

EXAMPLE I.

Fig. 65. To find the point of *Inflection* B in the
41. *Protracted Semicycloid* ABD; whose generating semicircle AGF is less than it's base FD*.

Put

* This Curve may be thus generated.—Let the semicircle *af* roll along upon the right line *fd* equal to it and perpendicular to it's diameter *fa*; then will the curve ABD described by any point A taken *within* the said semicircle and in the radius *Oa*, be a *Protracted Semicycloid*. For, describe the concentric semicircle AGF; in any position of which, as BRK, to the generating point B draw the ordinate CB parallel to the base

Put the generating semicircle $AGF=a$, base $FD=b$, radius $OG=c$, $CG=s$, $OC=x$, ordinate $CB=y$, and arch $AG=z$; and let the point g be supposed indefinitely near to G , and ng parallel to CF , that is, let $Gn=s'$, $ng=x'$, and $gG=z'$. Now, if the Increment Gg be supposed a little right line perpendicular to the radius OG , the right angled triangles OCG and Gng will be similar; and therefore, by 4 E. 6. $OC : CG :: Gn : ng$, that is, $x : s :: s' : x'$, or (*art. 7.*) $x : s :: \dot{s} : \dot{x}$, $\therefore \dot{s} = \frac{x\dot{x}}{s}$. By the known pro-

perty of the curve, $a : b :: z : GB = \frac{bz}{a}$; therefore $CG + GB = s + \frac{bz}{a} = y$; the Fluxion of which equation, because the Fluxions of s and z are negative to each other, is $-\dot{s} + \frac{b\dot{z}}{a} = \dot{y}$. Hence, $\dot{y} = -\frac{x\dot{x}}{s} + \frac{b\dot{z}}{a}$, that is, (because

by

base FD , which said base must evidently be equal and parallel to fd : through the center o draw Mr parallel to the diameter af ; and with the radius oB describe the arch BM : then will the arches AG , MB , and RK , be equal, and $oB=CG$; and therefore, $GB=Ct=fr$ = (by the generation) arch rk ; but, semicircle AF : semicircle af :: arch RK : arch rk ; therefore, semicircle AF : fd or base FD :: arch AG : GB ; which is the property of the *Cycloid*. (See *art. 35.*)

This Curve may, with propriety, be called an *interior Cycloid*.

by 47 E. 1. $s = \sqrt{c^2 - x^2}^{\frac{1}{2}}$, $\dot{y} = -\frac{x\dot{x}}{\sqrt{c^2 - x^2}^{\frac{1}{2}}} + \frac{b\dot{z}}{a}$. But, by 4 E. 6. $s : c :: x' : z'$; that is,

$$\sqrt{c^2 - x^2}^{\frac{1}{2}} : c :: \dot{x} : \dot{z} = \frac{c\dot{x}}{\sqrt{c^2 - x^2}^{\frac{1}{2}}}; \text{ wherefore,}$$

by substitution, $\dot{y} = \left(-\frac{x\dot{x}}{\sqrt{c^2 - x^2}^{\frac{1}{2}}} + \frac{bc\dot{x}}{a\sqrt{c^2 - x^2}^{\frac{1}{2}}} \right)$

$$=) \frac{-ax\dot{x} + bc\dot{x}}{a\sqrt{c^2 - x^2}^{\frac{1}{2}}}. \text{ Now, the Fluxion of this}$$

equation, making both $\ddot{x}$ and $\ddot{y} = 0$, is $0 =$

$$\frac{-a\dot{x}^2 \times a\sqrt{c^2 - x^2}^{\frac{1}{2}} + \frac{ax\dot{x}}{\sqrt{c^2 - x^2}^{\frac{1}{2}}} \times -ax\dot{x} + bc\dot{x}}{a^2\sqrt{c^2 - x^2}},$$

$$\text{that is, (by reduction,)} \quad 0 = \frac{-ac^2\dot{x}^2 + bcx\dot{x}^2}{a\sqrt{c^2 - x^2}^{\frac{1}{2}}};$$

whence, $0 = -ac + bx$, and therefore $x =$

$$\frac{ac}{b} = OC.$$

Or *, Put the generating semicircle AGF $= a$, base FD $= b$, radius OG $= c$, and OC $= x$; and let gb be supposed indefinitely near and parallel to GB. Now, it is evident, that, at the point of Inflection B, the angle made by the tangent and ordinate must be a Maximum; that is, at the said point, the

$\angle Bbv$

* Invented Anno 1760.

$\angle Bbv$ is a Maximum. But, by the nature or generation of the curve, $a : b :: Gg$ or $Bv : vb$; and, by Trigonometry, $bv : vB :: s.\angle vBb : s.\angle Bbv$; therefore, the said $\angle Bbv$ is the Greatest when vB is perpendicular to Bb . Consequently, at the point of Inflection B, the tangent is coincident with the radius oB ; and the triangles Bvb and ioB or COG are similar: therefore, by 4 E. 6. $Bv : vb :: CO : OG$; that is, $a : b :: x : c$; $\therefore x = \frac{ac}{b} = OC$; as before.

Corollaries.

1. At any point B in the curve, if to the corresponding point G in the circle, we draw the right line TG perpendicular to the radius OG, making $TG : GB :: a : b$, that is, if TG be made equal to the arch AG; or, if Bx be made equal to the radius Oa , and xz be drawn parallel to the tangent GT and equal to the radius OA; then will the right line drawn from T or z to B be a Tangent to the curve at the point B. For then the triangles TGB or zxB and Bvb will be similar; and therefore, &c.

2. If the right line AQ be drawn perpendicular to the radius Oa ; then, when the point

point *Q* arrives at the right line *fd* parallel to the base *FD*, the point *A* will be in the point of Inflection *B*. For, since at the said point of Inflection, $x = \frac{ac}{b}$; by ana-

logy, $b : a :: c : x$, that is, semicircle *prk* : semicircle *BRK* :: *oB* : *ot*; therefore, the semicircles being as their radii, $ro : oB :: Bo : ot$, that is, *ot* or *OC* is a third proportional to the radii *Of* and *OF*, and the triangles *roB* and *Bot* are similar. Consequently, *rB* is perpendicular to *po*, and therefore, &c.—Hence the following

Construction.

Make *DR* = arch *fe*; draw *rM* through the point *R* equal and parallel to *fA*; make *oR* = *OF*; and lastly, describe the semicircles *MR* and *or*: then will the intersecting point *B* of the said semicircles be the point of Inflection required.

EXAMPLE II.

66. To find the point of Inflection *B* in the Fig. Conchoid of Nicomedes *AB* &c.*

42. Put *PE* = *a*, *EA* = *b*, *EC* = *x*, and *CB* = *y*; then (*art.* 33.) $\dot{y} = \frac{-ab^2\dot{x} - x^3\dot{x}}{x^2 \times b^2 - x^2}^{\frac{1}{2}}$. Now, the

* See the generation of this Curve, *art.* 33. *note.*

the Fluxion of this equation, making both $\ddot{x}$ and $\ddot{y}=0$, is (by reduction) $0 = \frac{2ab^4x - 3ab^2x^3 - b^2x^4}{x^4 \times b^2 - x^2} \dot{x}^2$; therefore, $2ab^2 - 3ax^2 - x^3 = 0$; by which equation, x , and consequently the point B, may be determined.—And, if $a=b$, it will be $2a^3 - 3ax^2 - x^3 = 0$; which divided by $a+x$, makes $2a^2 - 2ax - x^2 = 0$; from which quadratic equation we have $x = \sqrt[3]{a^2} - a$.

Construction.

Make $Pn = \frac{2}{5}PE$; draw the indefinite right line nm perpendicular to Pn ; make $Es = \frac{1}{5}EA$; draw the right lines sr and Pm parallel to each other and perpendicular to the right line mr ; make $Et = EA$; and parallel to sr draw the right line tC : then an ordinate drawn from the point C will fall on the point of Inflection B. For, by 4 E. 6, $CE :$

$$Et :: Pn : nm, \text{ that is, } x : b :: \frac{2}{5}a : \frac{2ab}{5x} = nm;$$

$$\text{and } Pn : nm :: mn : nr, \text{ that is, } \frac{2}{5}a : \frac{2ab}{5x} :: \frac{2ab}{5x}$$

$$: \frac{2ab^2}{5x^2} = nr; \text{ again, } tE : EC :: sE : Er, \text{ that}$$

$$\text{is, } b : x :: \frac{1}{5}b : \frac{1}{5}x = Er; \text{ therefore, (} nE \text{ be-}$$

O

ing

ing $= \frac{1}{5}a$) $nr = \frac{1}{5}a + \frac{1}{5}x$. Hence, $\frac{2ab^2}{5x^2} = \frac{1}{5}a + \frac{1}{5}x$, and therefore $2ab^2 = 3ax^2 + x^3$, or $2ab^2 - 3ax^2 - x^3 = 0$.

Or,

When $a=b$; make $Pv=PA$, and $PC=Ev$: then will C be the point in the absciss from which the ordinate to the point of Inflection must be drawn. For then $\overline{Pv^2 - PE^2}^{\frac{1}{2}} = \overline{3a^2}^{\frac{1}{2}} = Ev = PC$; and therefore, $EC = \overline{3a^2}^{\frac{1}{2}} - a = x$.

EXAMPLE III.

67. To find the point of *Inflection* B in the
 Fig. Curve ABY, whose Equation (putting
 43. the absciss $AC=x$, ordinate $CB=y$, and the perpendicular $AE=a$,) is $ax^2 = a^2y + x^2y$.

The Fluxion of the equation of the curve is $2ax\dot{x} = a^2\dot{y} + 2x\dot{x}y + x^2\dot{y}$; therefore, $\dot{y} = \frac{2ax\dot{x} - 2x\dot{x}y}{a^2 + x^2}$, that is, (by writing $\frac{ax^2}{a^2 + x^2}$ for y it's value,) $\dot{y} = \frac{2a^3x\dot{x}}{a^2 + x^2}^2$; and the Fluxion of this equation, making both $\ddot{x}$ and $\ddot{y} = 0$, is $0 = \frac{2a^3\dot{x}^2 \times a^2 + x^2}^{2} - \frac{4a^2x\dot{x} + 4x^3\dot{x} \times 2a^3x\dot{x}}{a^2 + x^2}^4$;

which

which multiplied by $\sqrt{a^2 + x^2}$ and divided by $2a^3\dot{x}$, gives $0 = \sqrt{a^2 + x^2}^2 - 4a^2x^2 - 4x^4$; therefore $4x^4 + 4a^2x^2 = \sqrt{a^2 + x^2}^2$; which equation divided by $x^2 + a^2$, makes $4x^2 = \sqrt{a^2 + x^2}$; therefore, $3x^2 = a^2$, and $x = a\sqrt{\frac{1}{3}}$; and, if this value of x be substituted for it in the given equation of the curve, we shall have y , or, the ordinate at the point of Inflection $= \frac{1}{4}a$.

Construction.

Make $Ae = \frac{1}{3}AE$; and with the radius eE describe the arch EC : then will C be the point from which the ordinate to the point of Inflection must be drawn. For then $eC = \frac{2}{3}a$, and by 47E. 1. $Ce^2 - eA^2 = AC^2$, that is, $\frac{4}{9}a^2 - \frac{1}{9}a^2 = \frac{1}{3}a^2 = AC^2 = x^2$, or $x = a\sqrt{\frac{1}{3}}$.

Note.

If it were required to find the Asymptote to the curve:—Suppose the absciss and curve to be indefinitely extended: then, because x^2 will be indefinitely near to equality with $a^2 + x^2$, we shall have y (which by the equation of the curve is $= ax \frac{x^2}{a^2 + x^2}$), indefinitely near to equality with the given right line a ; that is, y will then be $= a$, minus a quantity

indefinitely small. Wherefore, if from the point E a right line, EZ, be drawn parallel to the axis, AX, it will be the Asymptote required.

SCHOLIUM.

68. IN any Curve, in order to know whether it be *concave* or *convex* towards any point assigned in the axis; find the value of $\ddot{y}$ at that point: then, (*art.* 61.) if this value of $\ddot{y}$ comes out Positive, the curve will be *convex* towards the axis; and if it comes out Negative, it will be *concave*.

Thus, in the last Example, if it were required to find whether the Curve be at first *concave* or *convex* towards the axis:—Suppose $a=10$, and make $x=1$, or $x=$ any number less than $a\sqrt{\frac{1}{3}}$ or $10\sqrt{\frac{1}{3}}$; then, because $\ddot{y}$ may here be considered to be always as $\frac{(a^2+x^2)^2-4a^2x^2-4x^4}{(a^2+x^2)^3}$, the expression for $\ddot{y}$ will come out Positive; and therefore, the curve is at first *convex* towards the axis: And, when $x=6$, or $x=$ any number greater than $a\sqrt{\frac{1}{3}}$ or $10\sqrt{\frac{1}{3}}$, the said expression will come out Negative; and therefore, then, the curve will be *concave* towards the axis.

C H A P. VI.

Of finding the Radius of Curvature.

69. AS the curvature, or convexity, of all curves, but circles, varies in every point; therefore, if circles are described coinciding with a given curve in any number of points, the Radii of these circles will be different: And the finding of these Radii is the business of this Chapter.

70. AND, because all curves, but circles, are formed or generated, or may be conceived to be formed or generated, by the evolution, or winding off, of some other curves; therefore, the centers of the circles which coincide or have equal degrees of curvature with the different Points or rather Increments of the curves thus formed or generated will always be in the curves to be unwound; which curves are called the *Evolutes*; and the others formed or generated, or conceived to be formed or generated, by their evolution, are called the *Involutes*.

71. Thus, let DEF be any curve, called *Fig.*
an Evolute; round which conceive a thread ^{44.}
to be wound and extended beyond the curve
from D in a right line to A; and let this
thread

thread be evolved, or wound off, from the curve DF, so that it be continually stretched at it's full length as it leaves the curve : then will the point A generate, or describe, the Involute curve ABY ; and the right lines AD, BE, YF, will be the Radii of Curvature at the points A, B, Y, respectively.

Corollaries.

1. The radius of curvature BE will always be equal to the length of the curve ED and right line DA : And, consequently, if the vertical distance, or shortest radius DA, vanishes ; that is, if the radius at A be nothing ; then, the Involute curve will begin at D ; and the curve DE will be equal to the radius of curvature at the point B.

2. Because (18 E. 3.) the radius of a circle is perpendicular to the tangent, the radius of curvature at any point B is always perpendicular to a tangent at that point.

3. The radius BE, which is perpendicular to the Involute at the point B, is a tangent to the Evolute at the point E.

P R O B L E M.

72. To deduce a General Expression for BE the Radius of Curvature at any point B in the Involute Curve ABY whose Axis is AX and Evolute DE.

Put the absciss $AC=x$, and ordinate $CB=y$: and suppose bE indefinitely near to BE , bc indefinitely near and parallel to BC ; and Bm parallel to AX ; that is, let Cc or $Bn=x'$, and $nb=y'$. Then, Bb being considered as a little right line coinciding with a tangent to the point B, the right angled triangles Bnb and BCH will be similar; (for $\angle Ebb=\angle CBn$, and therefore, the $\angle EBn$ being common, the $\angle s nBb$ and CBH are equal; *ergo*, &c.) and, the $\angle Bbm$ being right, the right angled triangles mnb and bnB will, by 8 E. 6, be similar also. Wherefore, by 4 E. 6, $Bn : nb :: BC : CH$, that is, $x' : y' :: y : \frac{yy'}{x'} = CH$; and therefore, by 47 E. 1.

$$BH = \sqrt{BC^2 + CH^2}^{\frac{1}{2}} = \sqrt{y^2 + \frac{y^2 y'^2}{x'^2}}^{\frac{1}{2}} = \frac{y}{x'} \times$$

$$\sqrt{x'^2 + y'^2}^{\frac{1}{2}}; \text{ also, } AH = x + \frac{yy'}{x'}: \text{ Again, } Bn$$

$$: nb :: bn : nm, \text{ that is, } x' : y' :: y' : nm = \frac{y'^2}{x'};$$

∴

$\therefore Bm = x' + \frac{y'^2}{x'}$. Now, because the Direc-

tion of the curve ABY approaches continually nearer to a Parallelism with the axis AX; therefore, if we suppose the absciss (AC=x,) to flow with an equable or uniform motion, that is, supposing x' or $\dot{x}$ to be invariable, or always of the same value; then, the Increment of the ordinate (CB=y,) or the Velocity or Fluxion with which it flows, must continually decrease; that is, the *Second Moment* or *Second Fluxion* of y will be negative: and therefore, Hb, the Increment of AH, viz. the Increment of $x + \frac{yy'}{x'}$,

will be $= x' + \frac{y'^2 - yy''}{x'}$. Now, the triangles

EBm and Ehb are evidently similar; therefore, Em—Hb: Bm :: (BE—HE=) BH: BE,

that is, $\frac{yy''}{x'} : x' + \frac{y'^2}{x'} :: \frac{y}{x'} \times \sqrt{x'^2 + y'^2}^{\frac{1}{2}} :$

$\frac{\sqrt{x'^2 + y'^2}^{\frac{1}{2}}}{x'y''} = BE$; or (*art. 7.*) $BE = \frac{\sqrt{x'^2 + y'^2}^{\frac{1}{2}}}{x'y''}$.

Or, With the radius EB describe the circular Arch BK; which Arch will therefore
 45. have the same Degree of Curvature with the Involute Curve AB at the point B. Draw the

the radius EK parallel to the axis AX; and produce the ordinate BC to L, to which draw AN parallel. Put the absciss AC= x , ordinate CB= y , radius EB or EK= r , KN= a , and NA= b ; then, LE= $r-a-x$. Now, if we suppose the absciss, x , to increase uniformly, and Bm to be a tangent to the point B; then, if we draw mn parallel to BC, and Bn parallel to AX, by art. 24. Bn, nm, and mB, will be as the Fluxions of the absciss, ordinate, and curve, respectively; that is, Bn will be as $\dot{x}$, nm as $\dot{y}$, and (because by 47 E. 1, $Bm = \sqrt{Bn^2 + nm^2}^{\frac{1}{2}}$,) Bm as $\sqrt{\dot{x}^2 + \dot{y}^2}^{\frac{1}{2}}$. Now, the triangles Bnm and BLE are similar; therefore, by 4 E. 6, Bn : nm :: BL : LE, that is, $\dot{x} : \dot{y} :: y + b : r - a - x$; $\therefore rx - ax - x\dot{x} = y\dot{y} + b\dot{y}$; the Fluxion of which equation (supposing $\dot{x}$ invariable, and therefore, the direction of the curve AB continually approaching towards a parallelism with it's axis, the Fluxion of $\dot{y}$ as negative,) is $-\dot{x}^2 = \dot{y}^2 - y\ddot{y} - b\ddot{y}$; $\therefore \dot{x}^2 + \dot{y}^2 = b + y \times \ddot{y}$. Again, by 4 E. 6, LB : BE :: nB : Bm, that is, $b + y : r :: \dot{x} : \sqrt{\dot{x}^2 + \dot{y}^2}^{\frac{1}{2}}$; $\therefore b + y = \frac{r\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}^{\frac{1}{2}}}$; which substituted for $b + y$ makes the above

P $\dot{x}^2$

$$\dot{x}^2 + \dot{y}^2 = \frac{r\ddot{x}\ddot{y}}{\dot{x}^2 + \dot{y}^2}^{\frac{1}{2}}; \text{ therefore, } \sqrt{\dot{x}^2 + \dot{y}^2} \times$$

$$\sqrt{\dot{x}^2 + \dot{y}^2}^{\frac{1}{2}} = r\ddot{x}\ddot{y}, \text{ that is, } \sqrt{\dot{x}^2 + \dot{y}^2}^{\frac{3}{2}} = r\ddot{x}\ddot{y}; \therefore$$

$$\frac{\sqrt{\dot{x}^2 + \dot{y}^2}^{\frac{3}{2}}}{\ddot{x}\ddot{y}} = r = \text{BE}; \text{ as before.}$$

73. *Note.* When the absciss, x , flows with an uniform motion; it follows from *art.* 61. that the ordinate, y , flows with a retarded motion when it increases and the curve is concave, or when it decreases and the curve is convex towards the axis; and with an accelerated motion when it decreases and the curve is concave, or when it increases and the curve is convex towards the axis. Now, when y increases with an accelerated motion, or it's *Second Fluxion* is affirmative, the General Expression for the Radius of Curvature will be $= \frac{\sqrt{\dot{x}^2 + \dot{y}^2}^{\frac{3}{2}}}{-\ddot{x}\ddot{y}}$; where the Negative sign shews it's position.

74. HENCE, because we may substitute 1 for any invariable Fluxion*; if we put

$$\dot{x} = 1,$$

* The substituting *unity*, or 1, for an invariable Fluxion, has no other effect than it's making the operation less laborious; and, in reality, it is no more than making *unity* the Standard of the other Fluxions, or reducing the other Fluxions to a comparison with 1.

$\dot{x}=1$, the General Expression for the Radius of Curvature will be $= \frac{\sqrt{1+\dot{y}^2}}{\ddot{y}}$ when y increases with a Retarded motion, or it's *Second*

Fluxion is Negative; and $= \frac{\sqrt{1+\dot{y}^2}}{-\ddot{y}}$ when y

increases with an Accelerated motion, or it's *Second Fluxion* is Affirmative. The former takes place when the curve is Concave, and the latter when it is Convex towards the axis.——Wherefore, if we put the Equation of the given curve, expressing the relation between the absciss x and ordinate y , into Fluxions, making $\dot{x}=1$; or, from the nature of the curve, find the value of $\dot{x}=1$ in terms of x , y , and $\dot{y}$; and then put this *fluxional* Equation into Fluxions again, still substituting 1 for $\dot{x}$, and making the Fluxion of $\dot{y}$ Negative when the curve is Concave, and Affirmative when it is Convex towards the axis; from thence the Values of the *second* and *square* of the *first Fluxions* of y may be determined: which being substituted for them in one of these two general expressions, viz. in the former when the Fluxion of $\dot{y}$ is Negative, and in the latter when it is Affirmative; we shall have a definitive expression

for BE, that is, an expression for it free from Fluxions, or, the Radius of Curvature required.

Note.

75. THE Vertical Distance, or Radius AD, may be obtained, by writing for $\dot{x}$ and $\dot{y}$ their values, in the General Expression for the Subnormal CH, which was found in *art.* 72.

($= \frac{yy'}{x'}$, that is, by writing $\dot{x}$ for x' , and $\dot{y}$

for y' .) $= \frac{yy}{x}$; or, making $\dot{x}=1$, by substituting the value of $\dot{y}$ in yy , that is, by multiplying the value of $\dot{y}$ by y ; and then making y vanish in the definitive expression which will then be found.—For, the expression for the Subnormal CH being the same at whatever point in the curve B is taken; therefore, if it be taken at A, where y vanishes or becomes $=0$, the point C must of consequence coincide with the vertex A, and the points E and H with D: therefore, &c.

EXAMPLE I.

Fig. 76. To find the Radius of Curvature at any point B in the *Parabola* AY.
46.

Put the parameter $=a$, absciss $AC=x$, and ordinate $CB=y$. Now, by a well known

known property of the curve, $ax=y^2$; the Fluxion of which equation is $a\dot{x}=2y\dot{y}$, or, making $\dot{x}=1$, it is $a=2y\dot{y}$; therefore, $\dot{y}=\frac{a}{2y}=\frac{a}{2\sqrt{ax}}^{\frac{1}{2}}$, for y is $=\sqrt{ax}^{\frac{1}{2}}$ by the equation

of the curve: And, the Fluxion of this equation again, (the direction of the curve approaching continually towards a parallelism with the axis, and therefore the Fluxion of $\dot{y}$ being negative, *art.* 61.) is $-\ddot{y}=\frac{-a^2}{4\sqrt{ax}}^{\frac{1}{2}}$: Whence, $\dot{y}^2=\frac{a^2}{4\sqrt{ax}}=\frac{a}{4x}$, and $\ddot{y}=\frac{-a^2}{4\sqrt{ax}}^{\frac{1}{2}}$.

Now, if for $\dot{y}^2$ and $\ddot{y}$ we substitute

these their values, we shall have $\frac{1+\dot{y}^2}{\ddot{y}}^{\frac{3}{2}}$, the general expression for the Radius of Curvature BE (*art.* 74.)

$$= \frac{\left(1 + \frac{a}{4x}\right)^{\frac{3}{2}} \times 4\sqrt{ax}}{a^2}$$

$$= \frac{(4ax + a^2)^{\frac{3}{2}}}{2a^2}$$

Construction.

Through the point B describe the semi-circle ABn; bisect Cn in H; make Hr=
2AC;

2AC; and drop the perpendicular rE , terminated by the right line BE drawn through the point H : then will BE be the Radius of Curvature at the point B . For, by 35 E. 3. $BC^2 = AC \times Cn$, or $\frac{BC^2}{AC} = Cn$, that is, $\frac{ax}{x} = a = Cn$, and therefore $CH = \frac{1}{2}a$, and $Cr = \frac{1}{2}a + 2x$: by 47 E. 1. $\overline{CH^2 + CB^2}^{\frac{1}{2}} = BH$, that is, $\overline{\frac{1}{4}a^2 + ax}^{\frac{1}{2}} = BH$; and by 4 E. 6. $CH : HB :: Cr : BE$, that is, $\frac{1}{2}a : \overline{\frac{1}{4}a^2 + ax}^{\frac{1}{2}} :: \frac{1}{2}a + 2x : \frac{a + 4x}{a} \times \overline{\frac{1}{4}a^2 + ax}^{\frac{1}{2}} = \frac{a^2 + 4ax}{2a^2} = BE$.

Note.

77. By writing for y it's value $\frac{a}{2y}$, in yy , (art. 75.) we have $\frac{a}{2}$ or $\frac{1}{2}a = AD$ the vertical Distance. Which same truth may be inferred from the expression for the Radius BE ; for, when the said Radius becomes the vertical Distance, that is, when the point B coincides with A , x vanishes; and therefore, by striking $4ax$ out of the said expression, we have $\frac{a^2}{2a^2} = \frac{1}{2}a$; as before.

Ex -

EXAMPLE II.

78. To find the Radius of Curvature at any *Fig.*
point B in the *Cycloid* ABD *. 47.

Put the radius OF or OD = a , absciss
AC = x , ordinate CB = y , fine IG = s , and
arch FG = z . Now, by 35 E. 3. $IG =$
 $\overline{DI \times IF}^{\frac{1}{2}}$, that is, $s = \overline{2ay - y^2}^{\frac{1}{2}}$; the Flu-
xion of which equation is $\dot{s} = \frac{a\dot{y} - y\dot{y}}{\overline{2ay - y^2}^{\frac{1}{2}}}$: And

by the nature of the Cycloid, (*art* 35.) arch
DG = GB; and therefore, arch FG = GI +
AC, or AC = arch FG - GI, that is, $x = z$
- $s =$ (by substituting for s it's above value,) $z - \overline{2ay - y^2}^{\frac{1}{2}}$; and the Fluxion of this equa-
tion, making $\dot{x} = 1$, is $1 = \dot{z} + \frac{y\dot{y} - a\dot{y}}{\overline{2ay - y^2}^{\frac{1}{2}}}$.

But (*art* 72.) $\dot{z} = \overline{\dot{s}^2 + \dot{y}^2}^{\frac{1}{2}} =$ (by writing for $\dot{s}$

it's above value,) $\overline{\frac{a\dot{y} - y\dot{y}}{\overline{2ay - y^2}^{\frac{1}{2}}} + \dot{y}^2}^{\frac{1}{2}} = \frac{a\dot{y}}{\overline{2ay - y^2}^{\frac{1}{2}}}$;

which substituted for $\dot{z}$ makes the above
equation $1 = \frac{a\dot{y}}{\overline{2ay - y^2}^{\frac{1}{2}}} + \frac{y\dot{y} - a\dot{y}}{\overline{2ay - y^2}^{\frac{1}{2}}}$; that is,

$$I =$$

* See how this Curve may be generated, *art* 35. *note*.

$$1 = \frac{y\ddot{y}}{(2ay - y^2)^{\frac{1}{2}}}; \text{ therefore } \dot{y} = \frac{(2ay - y^2)^{\frac{1}{2}}}{y}; \text{ and}$$

the Fluxion of this equation (the Fluxion of $\dot{y}$ being negative,) is $-\ddot{y} =$

$$\frac{ay\dot{y} - y^2\ddot{y}}{(2ay - y^2)^{\frac{1}{2}}} - \dot{y} \cdot \frac{-y}{(2ay - y^2)^{\frac{1}{2}}} = \frac{-a\dot{y}}{y \cdot (2ay - y^2)^{\frac{1}{2}}} = (\text{by}$$

writing for $\dot{y}$ it's equal,) $-\frac{a}{y^2}$; that is, $\ddot{y} =$

$\frac{a}{y^2}$. Now, by substituting $\frac{2ay - y^2}{y^2}$ for $\dot{y}^2$,

and $\frac{a}{y^2}$ for $\ddot{y}$, we have, by *art. 74.* $\frac{(1 + \dot{y}^2)^{\frac{3}{2}}}{\ddot{y}} =$

$$\frac{1 + \frac{2ay - y^2}{y^2}}{\frac{a}{y^2}} \times y^2 = \frac{(2ay)^{\frac{3}{2}}}{ay} = 2 \cdot (2ay)^{\frac{1}{2}} = \text{BE}$$

the Radius of Curvature required.

Construction.

Make $FH = GB$; and through the point H draw the right line BE , making $BH = HE = \text{chord } GF$: then will BE be the Radius of Curvature at the point B . For, *art. 35.* a tangent to the point B is parallel to the chord DG , and by *art. 71. corol. 2.* the radius

dius of curvature is always perpendicular to the tangent; therefore, because by 31 E. 3. the $\angle$ DGF is right, BE must be parallel to the chord GF. Now, by 4 and 8 E. 6. $DF:FG::GF:FI$ or CB, $\therefore GF = \sqrt{DF \times CB}^{\frac{x}{2}}$
 $= \sqrt{2ay}^{\frac{1}{2}}$, and $2GF = 2 \cdot \sqrt{2ay}^{\frac{1}{2}} = BE$.

Note.

79. By *art.* 75. if we multiply the value of $\dot{y}$, viz. $\frac{\sqrt{2ay-y^2}^{\frac{1}{2}}}{y}$, by y ; we shall have the

Subnormal $CH = \sqrt{2ay-y^2}^{\frac{1}{2}}$; which, when y vanishes, becomes $=0$, and equal to the vertical Distance: so that the Vertices of the Evolute and Involute curves coincide.

EXAMPLE III.

80. To find the Radius of Curvature at any *Fig.* point B in the Curve AD; whose nature ^{48.} is such, that, the Triangle CBT, made of the Ordinate, Tangent, and Subtangent, is always proportional to the Ordinate CB; or, whose Subtangent CT is equal to a given line $=a$. (*See art.* 36.)

Put $GC=x$, and $CB=y$; then, by *art.* 25. $\frac{\dot{x}y}{\dot{y}}=a$, that is, if $\dot{x}$ be made $=1$, $\frac{y}{\dot{y}} = a$;

Q

∴

$\therefore \dot{y} = \frac{y}{a}$; therefore, $\dot{y}^2 = \frac{y^2}{a^2}$, and (because here y flows with an accelerated motion, or, it's Second Fluxion is affirmative,) $\ddot{y} = \frac{\dot{y}}{a}$, that is, (by substituting $\frac{y}{a}$ for $\dot{y}$ it's value,) $\ddot{y} = \frac{y}{a^2}$. Now, by writing for $\dot{y}^2$ and $\ddot{y}$ these their values,

$$\text{in } \frac{\sqrt{1+\dot{y}^2}^{\frac{3}{2}}}{-\ddot{y}}, \text{ (art. 74.) we have } \frac{\sqrt{1+\frac{y^2}{a^2}}^{\frac{3}{2}} \times a^2}{-y} = \frac{\sqrt{a^2+y^2}^{\frac{3}{2}}}{-ay} = \text{the Radius of Curvature sought :}$$

Where the Negative sign, only shews, that, the Evolute and Radius of Curvature lie on the other side of the curve with regard to x and y .

Construction.

Draw Bn parallel to CT , Tn perpendicular to TB , nE perpendicular to nB , and BE parallel to Tn : then will BE be the Radius of Curvature at the point B . For, by 47 E. 1. $BT = \sqrt{TC^2 + CB^2}^{\frac{1}{2}} = \sqrt{a^2 + y^2}^{\frac{1}{2}}$, and by 4 E. 6. $CT : TB :: TB : Bn$, that is, $a : \sqrt{a^2 + y^2}^{\frac{1}{2}} :: \sqrt{a^2 + y^2}^{\frac{1}{2}} : \frac{a^2 + y^2}{a} = Bn$; and, $CB : BT :: nB : BE$,

; BE, that is, $y : \overline{a^2 + y^2}^{\frac{1}{2}} :: \frac{a^2 + y^2}{a} : \frac{\overline{a^2 + y^2}^{\frac{3}{2}}}{ay}$
 $= BE.$

EXAMPLE IV.

81. To find the Radius of Curvature at any *Fig.*
 point B in the Curve AD; whose nature 49.
 is such, that the Tangent BT is every-
 where equal to a given line $= a.$

Put $GC = x$, and $CB = y$; then, by 47 E. 1.
 $\overline{TB^2 - BC^2}^{\frac{1}{2}} = CT$, that is, $\overline{a^2 - y^2}^{\frac{1}{2}} = CT$
 $= (\text{art. 25.}) \frac{\dot{x}y}{\dot{y}}$; or, making $\dot{x} = 1$, $\overline{a^2 - y^2}^{\frac{1}{2}}$
 $= \frac{y}{\dot{y}}$; $\therefore \dot{y} = \frac{y}{\overline{a^2 - y^2}^{\frac{1}{2}}}$; therefore $\dot{y}^2 = \frac{y^2}{a^2 - y^2}$,
 and (because the Fluxion of y is affirmative,)

$$\ddot{y} = \frac{\dot{y} \times \overline{a^2 - y^2}^{\frac{1}{2}} + \frac{y^2 \dot{y}}{\overline{a^2 - y^2}^{\frac{1}{2}}}}{a^2 - y^2}, \text{ that is, by sub-}$$

stituting for $\dot{y}$ it's value, $\ddot{y} = \frac{y + \frac{y^3}{a^2 - y^2}}{a^2 - y^2} =$
 $\frac{a^2 y}{\overline{a^2 - y^2}^2}.$ Now, by writing for $\dot{y}^2$ and $\ddot{y}$

these their values, in $\frac{\overline{1 + \dot{y}^2}^{\frac{3}{2}}}{-\ddot{y}}$, (*art. 74.*) we
 Q₂ have

$$\text{have } \frac{\left(1 + \frac{y^2}{a^2 - y^2}\right)^{\frac{3}{2}} \times \overline{a^2 - y^2}^{\frac{1}{2}}}{-a^2 y} = \frac{\overline{a^2}^{\frac{3}{2}} \times \overline{a^2 - y^2}^{\frac{1}{2}}}{-a^2 y}$$

$$= -\frac{a}{y} \times \overline{a^2 - y^2}^{\frac{1}{2}} = \text{the Radius of Curvature required: Where the Negative sign}$$

shews it's position.

Construction.

On the extremity of the subtangent, T, erect the perpendicular TE; and draw the right line BE perpendicular to the tangent TB: then will BE be the Radius of Curvature at the point B; or, the point E will be in the Evolute curve. For, the triangles CBT and BTE will be similar; and therefore, by 4 E. 6. BC : CT :: TB : BE, that is, $y : \overline{a^2 - y^2}^{\frac{1}{2}} :: a : \frac{a}{y} \times \overline{a^2 - y^2}^{\frac{1}{2}} = \text{BE}.$

82. THE General Expression for the Radius of Curvature found in *art.* 74. being only for Curves referred to an Axis; we shall now deduce one for *Spirals*, or those Curves whose ordinates are referred to a fixed or central Point.

83. LET CBY be the Curve; C the central point; or, that from which all the ordinates issue; and BE the Radius of Curva-*Fig.* ture at the point B , that is, let the point E 50. be supposed in the Evolute curve: conceive Cb and Eb indefinitely near to CB and EB , that is, let the points B and b be supposed indefinitely near to each other; and let CF and Cf be perpendicular to EB and Eb respectively: then will the points F and r be indefinitely near to a coincidence; and therefore, *art.* 7. Br and Cr may be taken as equal to BF and CF . Now, if with the ordinate CB , as a radius, the little circular arch Bn be described and considered as a little right line perpendicular to Cb ; and the Increment Bb be considered as coinciding with a tangent to the point B ; then, the little right angled triangle Bnb will be similar to the right angled triangle BFC ; (for $\angle CBn = \angle EBb$; and therefore, $\angle EBn$ being common, the $\angle CBF = \angle nBb$; and consequently, the angles at F and n being right, $\angle BCF = \angle Bbn$;) therefore, by 4 E. 6. $bB : Bn :: CB : BF$; that is, (if we put the ordinate $CB = y$, $Bn = x'$, and $nb = y'$, when by 47 E. 1. Bb will

will be $= \overline{x'^2 + y'^2}^{\frac{1}{2}}, \overline{x'^2 + y'^2}^{\frac{1}{2}} : x' :: y :$

$\frac{x'y}{\overline{x'^2 + y'^2}^{\frac{1}{2}}} = BF \text{ or } Br ; \text{ And } Bb : bn :: BC$

$: CF, \text{ that is, } \overline{x'^2 + y'^2}^{\frac{1}{2}} : y' :: y : \frac{yy'}{\overline{x'^2 + y'^2}^{\frac{1}{2}}}$

$= CF \text{ or } Cr ; \text{ the Increment of which is } rf ; \text{ that is, (supposing } x' \text{ to be in-}$

$\overline{y'^2 + yy''} \times \overline{x'^2 + y'^2}^{\frac{1}{2}} - \frac{y'y'' \times yy'}{\overline{x'^2 + y'^2}^{\frac{1}{2}}}$
riable) $\frac{x'^2 + y'^2}{\overline{x'^2 + y'^2}^{\frac{3}{2}}} = rf. \text{ Again, the tri-}$

$= \frac{x'^2 y'^2 + y'^4 + yx'^2 y''}{\overline{x'^2 + y'^2}^{\frac{3}{2}}} = rf. \text{ Again, the tri-}$

angles EBb and Erf being fimilar, $Bb - rf :$
 $Bb :: (BE - rE, \text{ or }) rB : BE ; \text{ that is, } \overline{x'^2 + y'^2}^{\frac{1}{2}}$

$- \frac{x'^2 y'^2 + y'^4 + yx'^2 y''}{\overline{x'^2 + y'^2}^{\frac{3}{2}}} = \frac{x'^4 + x'^2 y'^2 - yx'^2 y''}{\overline{x'^2 + y'^2}^{\frac{3}{2}}}$

$: \overline{x'^2 + y'^2}^{\frac{1}{2}} :: \frac{x'y}{\overline{x'^2 + y'^2}^{\frac{1}{2}}} : \frac{y \times \overline{x'^2 + y'^2}^{\frac{3}{2}}}{x'^3 + x'y'^2 - yx'y''}$

$= BE ; \text{ or, art. 7. } BE = \frac{y \times \overline{x'^2 + y'^2}^{\frac{3}{2}}}{x'^3 + x'y'^2 - yx'y''} ; \text{ which}$

is a General Expreffion for the Radius of Curvature of all Curves referred to a fixed or central Point, when x' or $\dot{x}$ is inva-
riable.

84. HENCE, if $\dot{x}$ be made $=1$, the General Expresssion for the Radius of Curvature will be $= \frac{y \times \sqrt{1 + \dot{y}^2}^{\frac{3}{2}}}{1 + \dot{y}^2 - y\ddot{y}}$. — Where-

fore, if we put the Equation of the given Spiral into Fluxions, (making $\dot{x}=1$.) and put this *fluxional* Equation into Fluxions again; and from thence, or from the nature of the curve, find the values of $\dot{y}^2$ and $\ddot{y}$: then, if for $\dot{y}^2$ and $\ddot{y}$ we substitute these their values, in this General Expresssion, we shall have BE the Radius of Curvature required: As in the following Examples.

EXAMPLE I.

85. To find the Radius of Curvature at *Fig.* any point B in the *Spiral of Archimedes*, 51. CB &c*.

Put the circumference of the generating circle AF &c. $=a$, and it's radius CA $=b$; ordinate CB $=y$, arch AF $=z$. Let Cf be supposed indefinitely near to CF, that is, let the $\angle FCf$ be supposed indefinitely small; and with the ordinate CB, as a radius, describe the little circular arch Bn, which put $=x'$;

* See how this Curve is generated, *art.* 39. *note.*

$=x'$; also, put $Ff=z'$. Now, by the nature of the curve, $a : b :: z : y$, or $z = \frac{ay}{b}$,

the Fluxion of which equation is $\dot{z} = \frac{a\dot{y}}{b}$:

And, by the similar sectors CBn and CFf , $y : x' :: b : z' = \frac{bx'}{y}$, or, *art. 7.* $\dot{z} = \frac{b\dot{x}}{y}$. Hence

$\frac{a\dot{y}}{b} = \frac{b\dot{x}}{y}$; that is, (making $\dot{x}=1$,) $\frac{a\dot{y}}{b} = \frac{b}{y}$;

from which equation we have $\dot{y} = \frac{b^2}{ay}$; there-

fore $\dot{y}^2 = \frac{b^4}{a^2y^2}$, and $\ddot{y} = \frac{-ab^2\dot{y}}{a^2y^2} =$ (by writ-

ing for $\dot{y}$ it's value,) $\frac{-b^4}{a^2y^3}$: And, if we

substitute for $\dot{y}^2$ and $\ddot{y}$ these their va-

lues, we shall have $\frac{y \times 1 + \dot{y}^2}{1 + \dot{y}^2 - y\ddot{y}}^{\frac{3}{2}}$ (*art. 84.*) =

$$\frac{\left(y \times 1 + \frac{b^4}{a^2y^2}\right)^{\frac{3}{2}}}{1 + \frac{b^4}{a^2y^2} + \frac{b^4}{a^2y^2}} = \frac{\left(a^2y^2 + b^4\right)^{\frac{3}{2}}}{a^3y^2 + 2ab^4} = \text{BE, the Ra-}$$

dus of Curvature sought.

Con-

Construction.

Through the center C draw the indefinite right line Hv perpendicular to the ordinate CB; draw the tangent TB, perpendicular to which draw BH; produce BC to V, making $BR=TH$, and $RV=CH$; with BV and BR, as radii, describe the arches Vv and Rr; draw the right line vB; and from the intersecting point r draw rE parallel to vH: then will BE be the Radius of Curvature at the point B.

B. For, (*art.* 39.) $CT=\frac{yz}{b}$, that is, by

substituting $\frac{ay}{b}$ for z , $CT=\frac{ay^2}{b^2}$; and by 3.

and 4 E. 6. $TC : CB :: CB : CH$, that is,

$\frac{ay^2}{b^2} : y :: y : \frac{b^2}{a} = CH$; therefore $TH=BR$

$=Br=\frac{ay^2}{b^2}+\frac{b^2}{a}$, and $BV=Bv=\frac{ay^2}{b^2}+\frac{2b^2}{a}$,

and, by 47 E. 1. $HB=\sqrt{BC^2+CH^2}^{\frac{1}{2}}=$

$\sqrt{y^2+\frac{b^4}{a^2}}^{\frac{1}{2}}=\frac{a^2y^2+b^4}{a}^{\frac{1}{2}}$: Again, by 4 E. 6.

$Bv : BH :: Br : BE$, that is, $\frac{ay^2}{b^2} + \frac{2b^2}{a} :$

R

$$\frac{a^2y^2+b^4}{a}^{\frac{1}{2}}$$

$$\frac{a^2 y^2 + b^4}{a} \div \frac{ay^2}{b^2} + \frac{b^2}{a} \cdot \frac{a^2 y^2 + b^4}{a} \times \frac{a^2 y^2 + b^4}{a^2 y^2 + 2b^4}$$

$$= \frac{a^2 y^2 + b^4}{a^2 y^2 + 2ab^4} = BE.$$

EXAMPLE H.

Fig. 86. To find the Radius of Curvature at any point B in the *Logarithmic Spiral* CBY; whose Equation (putting the ordinate CB = y, curve CB = x, and a and b for two given quantities,) is $ax = by$. (See *art. 40.*)

The Fluxion of the equation of the curve is $az = by$; therefore $\dot{z} = \frac{by}{a}$. Let the angle BCb be supposed indefinitely small; and with the ordinate CB, as a radius, let the little circular arch Bn be described. Now, if we consider Bn as a little right line perpendicular to Cb, and Bb as a little right line coinciding with a tangent to the point B; then, by 47 E. I. $Bb = \overline{Bn^2 + nb^2}^{\frac{1}{2}}$, that is, (putting Bn = x', nb = y', and Bb = z',) $z' = \overline{x'^2 + y'^2}^{\frac{1}{2}}$, or, by substituting the Fluxion for the Increment, $\dot{z} = \overline{\dot{x}^2 + \dot{y}^2}^{\frac{1}{2}}$, that is, (if we put $\dot{x} = 1$,) $\dot{z} = \overline{1 + \dot{y}^2}^{\frac{1}{2}}$. Hence $\frac{by}{a}$

$\frac{by}{a} = \sqrt{1+y^2}^{\frac{1}{2}}$; which equation squared is $\frac{b^2 y^2}{a^2}$

$= 1+y^2$; and this produces $y^2 = \frac{a^2}{b^2 - a^2}$;

therefore, $y = \frac{a}{\sqrt{b^2 - a^2}^{\frac{1}{2}}}$; and, this being an

invariable quantity, therefore $\ddot{y} = 0$. Now, by writing for y^2 and $\ddot{y}$ these their values, the general expression for the Radius of Cur-

vature, viz. $\frac{y \times \sqrt{1+y^2}^{\frac{3}{2}}}{1+y^2 - y\ddot{y}}$ art. 84. will become =

$$\frac{y \times \sqrt{1 + \frac{a^2}{b^2 - a^2}}^{\frac{3}{2}}}{1 + \frac{a^2}{b^2 - a^2} - 0} = y \times \frac{\sqrt{\frac{b^2}{b^2 - a^2}}^{\frac{1}{2}}}{\frac{b^2}{b^2 - a^2}} = y \times \frac{b}{\sqrt{b^2 - a^2}^{\frac{1}{2}}}$$

= BE, the Radius of Curvature required.

Construction.

Draw the tangent TB; perpendicular to which draw the right line BE, terminated by the subtangent TC produced: then will the point E be in the Evolute curve; or, the right line BE will be the Radius of Curvature at the point B. For then (TE being perpendicular to the ordinate CB,) by 8 E. 6. the triangles CTB and CBE will be similar;

R 2

and

and therefore, by 4 E. 6. $CT : TB :: CB : BE$; that is, (because by *art.* 40. $CT : TB :: \sqrt{b^2 - a^2}^{\frac{1}{2}} : b$), $\sqrt{b^2 - a^2}^{\frac{1}{2}} : b :: y : y \times \frac{b}{\sqrt{b^2 - a^2}^{\frac{1}{2}}} = BE$.

CH A P. VII.

Of finding the Nature of the Evolute of a given Involute Curve.

AS it is absolutely necessary for the Learner to be well acquainted with the foregoing Chapter before he enters upon this; we shall not here define the meaning of Evolute and Involute curves, it being sufficiently explained therein.

Fig. 87. LET BE be the Radius of Evolution 53. (or Curvature) at any point B in the Involute curve AB, whose absciss is $AC = x$, and ordinate $CB = y$. Parallel to HA draw EN; produce BC to L; and, equal and parallel to CL, draw DN from the vertex of the Evolute DE. Then will the triangles BHC and BEL be similar; and therefore, by 4 E. 6. $BH : HC :: BE : EL$, that is, (by *art.* 72

and 7.) $\frac{y}{x} \times \sqrt{x^2 + y^2}^{\frac{1}{2}} : \frac{y\dot{y}}{x} :: \frac{\dot{x}^2 + \dot{y}^2}{\dot{x}\dot{y}}^{\frac{1}{2}} : y \times \frac{\dot{x}^2 + \dot{y}^2}{\dot{x}\dot{y}} = EL$; and $HC : CB :: EL : LB$,

that

that is, $\frac{y\dot{y}}{\dot{x}} : y :: \dot{y} \times \frac{\dot{x}^2 + \dot{y}^2}{\dot{x}\dot{y}} : \frac{\dot{x}^2 + \dot{y}^2}{\dot{y}} = \text{LB.}$

Now, these are General Expressions for EL and LB, when $\dot{x}$ is considered as invariable, and the Fluxion of $\dot{y}$ as negative. Hence therefore,

88. If $\dot{x} = 1$, and the Fluxion of $\dot{y}$ be negative; the General Expression for BL will be $= \frac{1 + \dot{y}^2}{\dot{y}}$, and this multiplied by $\dot{y}$

is $\dot{y} \times \frac{1 + \dot{y}^2}{\dot{y}} =$ the General Expression for LE. Now, by help of the Equation of the given Involute curve, exterminate $\dot{y}$, $\dot{y}^2$, and $\dot{y}$, out of these expressions, as in the preceding Chapter; and, by *art.* 75. find the vertical distance AD. Then, if we put the absciss of the Evolute DN $= u$, and it's ordinate NE $= v$; by help of these two equations, $u = \text{BL} - \text{BC}$, and $v = \text{AC} - \text{AD} + \text{LE}$, we may get the Nature of the Evolute curve DE required.

89. *Note.* If the given Involute be convex towards it's axis, and x and y increase together, or the Fluxions of x and y be both affirmative; then, the General Expressions for BL and LE will be $\frac{1 + \dot{y}^2}{-\dot{y}}$ and $\dot{y} \times \frac{1 + \dot{y}^2}{-\dot{y}}$
re-

respectively; wherein, the Negative sign shews, that, the points L and E must be taken on the Concave side of the Involute curve, that is, on the other side of it with regard to x and y .

EXAMPLE I.

90. To find the Nature of the Curve DE, by whose Evolution the *Parabola* AB is described.

Put $AC=x$, $CB=y$; and $DN=u$, $NE=v$. Now, (by art. 76.) $\dot{y} = \frac{a}{2 \times \sqrt{ax}}^{\frac{1}{2}}$, $\dot{y}^2 =$

$\frac{a}{4x}$, and $\ddot{y} = -\frac{a^2}{4 \times \sqrt{ax}}^{\frac{3}{2}}$; which values of $\dot{y}$, $\dot{y}^2$,

and $\ddot{y}$, being substituted for them, make the General Expression for BL, viz. $\frac{1 + \dot{y}^2}{\ddot{y}}$ (art.

88.) become $= \frac{1 + \frac{a}{4x} \times 4 \times \sqrt{ax}}^{\frac{3}{2}} = \frac{\sqrt{4x + a \times ax}}^{\frac{1}{2}} \div \frac{a}{4x}$;

and that for LE, viz. $\dot{y} \times \frac{1 + \dot{y}^2}{\ddot{y}} = \dot{y} \times BL =$

$\frac{a}{2 \times \sqrt{ax}}^{\frac{1}{2}} \times \frac{\sqrt{4x + a \times ax}}^{\frac{1}{2}} = 2x + \frac{1}{2}a$. Hence, u

(=

$$(\text{BL} - \text{BC}) = \frac{4x + a\sqrt{ax}}{a} \quad (y \text{ or } \sqrt{ax})^{\frac{1}{2}}$$

$$= \frac{4x\sqrt{ax}}{a}; \text{ and (because by art. 77. the}$$

vertical distance AD is $= \frac{1}{2}a$), $v (= \text{AC} - \text{AD} + \text{LE}) = x - \frac{1}{2}a + 2x + \frac{1}{2}a = 3x$. Now, the former of these two equations produces

$$x^3 = \frac{au^2}{16}; \text{ and, the cube of the latter, di-}$$

$$\text{vided by 27, is } x^3 = \frac{v^3}{27}; \text{ therefore, } \frac{au^2}{16} =$$

$$\frac{v^3}{27}, \text{ and } \frac{27a}{16}u^2 = v^3; \text{ which is the equation}$$

of the Curve DE, expressing the relation between the absciss and ordinate; and, the Equation of the Semicubical Parabola

(whose parameter is $\frac{27a}{16}$), being the same;

therefore, the Evolute DE is a *Semicubical Parabola*, whose vertex is D.

Ex-

EXAMPLE II.

Fig. 91. To find the Nature of the Curve AEP;
 54. by whose Evolution the *Cycloid* ABD is described *.

Put $AC=x$, $CB=y$, arch $FG=z$, and
 OD or OF= a ; then (art. 78.) $\dot{y} = \frac{\sqrt{2ay-y^2}}{y}$,

$\ddot{y} = \frac{2ay-y^2}{y^2}$, and $\ddot{y} = \frac{a}{y^2}$: wherefore, (art.

$$88.) BL = \frac{1+\ddot{y}}{\ddot{y}} = \frac{1 + \frac{2ay-y^2}{y^2} \times y^2}{a} = 2y,$$

$$\text{and } LE = \dot{y} \times BL = \frac{\sqrt{2ay-y^2}}{y} \times 2y =$$

$\frac{2\sqrt{2ay-y^2}}{y}$. Hence, if we put the absciss
 AN= u , and ordinate NE= v , we have u
 ($=BL-CB=$) $2y-y=y$, and $v= (AC+LE=)$
 $x + \frac{2\sqrt{2ay-y^2}}{y}$, that is, (because,
 art. 78. $x=z-\sqrt{2ay-y^2}$), $v=z + \sqrt{2ay-y^2}$,
 or, (writing u for y it's equal,) $v=z + \sqrt{2au-u^2}$. Wherefore, the Evolute curve
 AEP is a *Cycloid*, and equal to the given
 Cy-

* See in what manner a *Cycloid* is generated, art. 35. note.

Cycloid ABD. For, let $AS=SV=a$; then (AN being $=FI$.) $AT=FG=z$, and $NT = \sqrt{2au-u^2}^{\frac{1}{2}} = IG$; and therefore $AT+TN = z + \sqrt{2au-u^2}^{\frac{1}{2}}$, that is, $AT+TN=NE$; which is the property of the Cycloid: therefore, the Evolute AEP is a Cycloid; and, because $AV=FD$, therefore the Cycloids AEP and ABD are equal.

EXAMPLE III.

92. To find the Nature of the Evolute of *Fig.* the Curve AD, whose Tangent BT is 55. every-where equal to a given line $=a$.

Let BE be the Radius of Curvature at the point B; then (*art.* 81.) if a perpendicular be erected on the point T, it will pass through the point E; wherefore, when the point T coincides with F, that is, when the tangent and ordinate become equal, or the points B and D coincide, the point E will likewise coincide with D: consequently, the Vertex of the Evolute coincides with that of the Involute.

Put $GF=b$, $GC=x$, $CB=y$, $DN=u$, and $NE=v$; then, (*art.* 81.) $y = \frac{y}{a^2-y^2}^{\frac{1}{2}}$
 $\dot{y} =$

$$\dot{y}^2 = \frac{y^2}{a^2 - y^2}, \text{ and } \ddot{y} = \frac{a^2 y}{(a^2 - y^2)^{\frac{3}{2}}}: \text{ wherefore,}$$

$$(\text{art. 89.}) \text{ BL} = \frac{1 + \dot{y}^2}{-\ddot{y}} = 1 + \frac{y^2}{a^2 - y^2} \times -$$

$$\frac{(a^2 - y^2)^{\frac{3}{2}}}{a^2 y} = \frac{a^2 - y^2}{-y}, \text{ and LE} = \dot{y} \times \text{BL} =$$

$$\frac{y}{(a^2 - y^2)^{\frac{1}{2}}} \times \frac{a^2 - y^2}{-y} = -\sqrt{a^2 - y^2}^{\frac{1}{2}}; \text{ that is,}$$

(because the Negative sign only shews that the points L and E must be taken on the Concave side of the Involute curve DA,)

$$\text{BL} = \frac{a^2 - y^2}{y}, \text{ and LE} = \sqrt{a^2 - y^2}^{\frac{1}{2}}. \text{ Hence}$$

$$\text{we have } u = (\text{LB} + \text{BC} - \text{DF}) = \frac{a^2 - y^2}{y} +$$

$$y - a, \text{ which equation gives } y = \frac{a^2}{u + a}, \text{ and}$$

$$\text{therefore } \dot{y} = -\frac{a^2 \dot{u}}{(u + a)^2}; \text{ also, } v = (\text{GF} -$$

$$\text{GC} + \text{CT}) = b - x + \sqrt{a^2 - y^2}^{\frac{1}{2}}, \text{ and there-}$$

$$\text{fore } \dot{v} = -\dot{x} - \frac{y \dot{y}}{\sqrt{a^2 - y^2}^{\frac{1}{2}}} = (\text{because by art. 81.}$$

$$\text{CT} = \frac{\dot{x} y}{\dot{y}} = \sqrt{a^2 - y^2}^{\frac{1}{2}}, \text{ or, } \dot{x} = \frac{\dot{y}}{y} \times \sqrt{a^2 - y^2}^{\frac{1}{2}},)$$

$$\frac{\dot{y} \times \overline{a^2 - y^2}^{\frac{1}{2}}}{y} - \frac{y\dot{y}}{\overline{a^2 - y^2}^{\frac{1}{2}}} = \frac{-a^2\dot{y}}{y \times \overline{a^2 - y^2}^{\frac{1}{2}}}, \text{ that}$$

is, (by writing for y and $\dot{y}$ their above values affected with u and $\dot{u}$), $\dot{v} = \frac{a\dot{u}}{\overline{u^2 + 2au}^{\frac{1}{2}}}$;

which is an Equation for the Evolute DE, and is also an Equation of the *Catenary* curve: therefore, the Evolute DE is the *Catenary* *.

Or, The above Equation of the Evolute may be found thus.—Let $En = u'$, and $ne = v'$; then, the triangles enE and TBE being similar, we have, by 4 E. 6. $en : nE :: TB : BE$, that is, $v' : u' :: a : \frac{au'}{v'} = BE$, or (*art. 7.*)

$$\frac{a\dot{u}}{\dot{v}} = BE; \text{ but, by 47 E. 1. } BE = \overline{ET^2 - TB^2}^{\frac{1}{2}},$$

that is, (because $ET = NF = u + a$), $BE = \overline{(u+a)^2 - a^2}^{\frac{1}{2}} = \overline{u^2 + 2au}^{\frac{1}{2}}$: therefore, $\frac{a\dot{u}}{\dot{v}} =$

$$\overline{u^2 + 2au}^{\frac{1}{2}}, \text{ and } \dot{v} = \frac{a\dot{u}}{\overline{u^2 + 2au}^{\frac{1}{2}}}; \text{ as before.}$$

SCHÖ-

* The *Catenary* is a Curve, as ADB or adb , formed by a flexible line or chain hanging freely from two points of suspension, A and B , or a and b , whether the said points be horizontal or not.

Fig.

56.

SCHOLIUM.

93. THE Evolute of a *Spiral*, or indeed of any other Curve, may be described, by finding the Radii of Curvature at several points in the Involute: For then we shall have as many points in the Evolute; through which, if a Curve line be drawn, it will be the Evolute sought.

P A R T II.

C H A P. I.

Of Infinite Series *.

AS the Learner may, perhaps, be unacquainted with *Infinite Series*, the knowledge of which is sometimes absolutely necessary in order to find the *Fluents* (or flowing quantities) of *Fluxions* expressed in a Fractional manner, and of such wherein there are Surds or Radical quantities; and, because in some of the following pages the *Fluents* of such *Fluxional* expressions are to be found; the adding of this Chapter may therefore not be improper, though it is, in some measure, foreign to the business in hand.

P R O B.

* The Methods of reducing compound expressions into *Infinite Series*, by division and extracting of roots, as taught in this Chapter, were invented by the great Inventor of *Fluxions* about the year 1664; who at the same time, or rather a little before, invented the celebrated *Binomial Theorem*.

P R O B. I.

To reduce a compound Fractional expression into an *Infinite Series*; that is, into a number of terms, which, if infinitely continued, shall be equal to the given fractional expression.

E X A M P L E . I.

94. To reduce $\frac{b}{a+x}$ into an *Infinite Series*.

Place the denominator $a+x$ as a divisor, and the numerator b as a dividend; and divide, as in common algebraic division, until you have 4, 5, 6, or more terms in the Quotient; after which you may find as many terms as you please, by only considering the *law* of the progression of the terms already found. Thus, the four first terms being $\frac{b}{a} - \frac{bx}{a^2} + \frac{bx^2}{a^3} - \frac{bx^3}{a^4}$, (see the operation below,) the *law* of the continuation of the division, or of the *Series*, is plain; for each Succeeding term is evidently produced by multiplying the Preceding by $-\frac{x}{a}$; and consequently, the fifth term will be $+\frac{bx^4}{a^5}$, the sixth term $-\frac{bx^5}{a^6}$, &c.

Ope-

Operation.

$$a+x) b \dots \left(\frac{b}{a} - \frac{bx}{a^2} + \frac{bx^2}{a^3} - \frac{bx^3}{a^4} + \mathcal{E}c. \right)$$

$$\begin{array}{r}
 b + \frac{bx}{a} \\
 \hline
 \circ - \frac{bx}{a} \\
 \frac{bx}{a} - \frac{bx^2}{a^2} \\
 \hline
 \circ + \frac{bx^2}{a^2} \\
 \frac{bx^2}{a^2} + \frac{bx^3}{a^3} \\
 \hline
 \circ - \frac{bx^3}{a^3} \\
 \frac{bx^3}{a^3} - \frac{bx^4}{a^4} \\
 \hline
 \circ + \frac{bx^4}{a^4} \\
 \mathcal{E}c.
 \end{array}$$

Or, If we put x before a , in the denominator of the above fractional expression; that is, if the divisor be placed thus, $x+a$, instead of $a+x$; then the Quotient, or Series, will be $\frac{b}{x} - \frac{ba}{x^2} + \frac{ba^2}{x^3} - \frac{ba^3}{x^4} + \mathcal{E}c.$

Whence,

Whence, the *law* of the continuation of the *Series* may be observed as before.

SCHOLIUM.

95. IN General, in order to have a *true* or *converging* Series, or that in which the terms continually Decrease, the Greatest term must be placed first. Thus, in the above Example, if a be greater than x ; then a must be the first term in the divisor, and $\frac{b}{a} - \frac{bx}{a^2} + \frac{bx^2}{a^3} - \frac{bx^3}{a^4} + \&c.$ will be the *true* Series: But, if x be greater than a ; then x must be the first term in the divisor, and $\frac{b}{x} - \frac{ba}{x^2} + \frac{ba^2}{x^3} - \frac{ba^3}{x^4} + \&c.$ will be the *true* Series; the other, then, being a *diverging* one, the terms in it continually Increasing; and consequently the farther you go in the *Series* the farther it will be from the truth.

THOUGH it is impossible to take any number of terms in the *Series* that shall *truly* express the value of the quantity given; yet, in general, a few of the leading terms will be near enough the truth for any purpose.

Ex-

EXAMPLE II.

96. To reduce $\frac{a^4}{a^2+2ax+x^2}$ into an *Infinite Series*.

Operation.

$$\begin{array}{r}
 a^2+2ax+x^2 \overline{) a^4 \dots (1 - \frac{2x}{a} + \frac{3x^2}{a^2} - \frac{4x^3}{a^3} + \text{&c.} \\
 \underline{a^2+2ax+x^2} \\
 0 - 2ax - x^2 \\
 \underline{-2ax - 4x^2 - \frac{2x^3}{a}} \\
 0 + 3x^2 + \frac{2x^3}{a} \\
 \underline{3x^2 + \frac{6x^3}{a} + \frac{3x^4}{a^2}} \\
 0 - \frac{4x^3}{a} - \frac{3x^4}{a^2} \\
 \underline{-\frac{4x^3}{a} - \frac{8x^4}{a^2} - \frac{4x^5}{a^3}} \\
 0 + \frac{5x^4}{a^2} + \frac{4x^5}{a^3}
 \end{array}$$

Now, from these four terms of the series it is easy to see the *law* of the continuation is such, that, the numerators are the powers

T

of

of x , whose Indices are 1 less than the numbers of the terms to which they respectively belong, multiplied by the said numbers; that, the denominators are the powers of a , whose Indices are the same with those of the numerators; and that, the signs of the terms are alternately changed. So that, the 5th term is $+\frac{5x^4}{a^4}$, the 6th term is $-\frac{6x^5}{a^5}$; and so on.

P R O B. II.

To reduce a compound Surd quantity into an *Infinite Series*; that is, to free a compound expression from Surds by throwing it into a number of decreasing terms, which, if infinitely continued, shall be equal to the quantity given.

E X A M P L E I.

97. To reduce $\sqrt{a^2+4y^2}^{\frac{1}{2}}$ into an *Infinite Series*.

Take the square root of a^2 , which is a , for the First term of the *Series*; (see the Operation below;) then, this squared and subtracted from a^2+4y^2 , leaves $4y^2$; and this remainder divided by the double of the first term,

term, (as in the common arithmetical extraction of the square root,) viz. by $2a$, gives

$+\frac{2y^2}{a}$ for the Second term of the Series;

which, with the double of the first term, be-

ing multiplied by $\frac{2y^2}{a}$ the said second term,

gives $4y^2 + \frac{4y^4}{a^2}$, and this subtracted from

$4y^2$ leaves $-\frac{4y^4}{a^2}$, which divided by the dou-

ble of the two first terms of the Series, viz.

by $2a + \frac{4y^2}{a}$, gives $-\frac{2y^4}{a^3}$ for the Third term

of the Series; which, with the double of the

two first terms, viz. $2a + \frac{4y^2}{a}$, being multi-

plied by $-\frac{2y^4}{a^3}$ the said third term, gives $-\frac{4y^4}{a^2}$

$-\frac{8y^6}{a^4} + \frac{4y^8}{a^6}$, and this subtracted from $-\frac{4y^4}{a^2}$

leaves $\frac{8y^6}{a^4} - \frac{4y^8}{a^6}$, which divided by the double

of the three first terms of the Series already

found, viz. by $2a + \frac{4y^2}{a} - \frac{4y^4}{a^3}$, gives $+\frac{4y^6}{a^5}$

for the Fourth term of the Series.—After

the same manner may be found any number

of terms in the Series: And, when the *law* of the progression, or of the continuation of the Series, is discovered, the terms may be continued on at pleasure.

Operation.

$$\begin{array}{r}
 a^2 + 4y^2 \left(a + \frac{2y^2}{a} - \frac{2y^4}{a^3} + \frac{4y^6}{a^5} - \text{&c.} \right) \\
 \hline
 2a \left(0 + 4y^2 \right) \\
 \hline
 4y^2 + \frac{4y^4}{a^2} \\
 \hline
 2a + \frac{4y^2}{a} \left(0 - \frac{4y^4}{a^2} \right) \\
 \hline
 -\frac{4y^4}{a^2} - \frac{8y^6}{a^4} + \frac{4y^8}{a^6} \\
 \hline
 2a + \frac{4y^2}{a} \left(0 + \frac{8y^6}{a^4} - \frac{4y^8}{a^6} \right) \\
 \hline
 \frac{8y^6}{a^4} + \frac{16y^8}{a^6} - \frac{16y^{10}}{a^8} + \frac{16y^{12}}{a^{10}} \\
 \hline
 0 - \frac{20y^8}{a^6} + \frac{16y^{10}}{a^8} - \frac{16y^{12}}{a^{10}}
 \end{array}$$

Ex.

EXAMPLE II.

98. To reduce $\sqrt{1-x-x^2}$ into an *Infinite Series*.

Operation.

$$1-x-x^2 \left(1 - \frac{x}{2} - \frac{5x^2}{8} - \frac{5x^3}{16} - \&c. \right)$$

$$\begin{array}{r}
 \overset{1}{2) 0-x-x^2} \\
 \underline{-x+\frac{x^2}{4}} \\
 2-x) 0-\frac{5x^2}{4} \\
 \underline{-\frac{5x^2}{4}+\frac{5x^3}{8}+\frac{25x^4}{64}} \\
 2-x-\frac{5x^2}{4}) 0-\frac{5x^3}{8}-\frac{25x^4}{64} \\
 \underline{-\frac{5x^3}{8}+\frac{5x^4}{16}+\frac{25x^5}{64}+\frac{25x^6}{256}} \\
 0-\frac{45x^4}{64}-\frac{25x^5}{64}-\frac{25x^6}{256} \\
 \&c.
 \end{array}$$

So that $\sqrt{1-x-x^2}$ is $= 1 - \frac{x}{2} - \frac{5x^2}{8} - \frac{5x^3}{16} - \&c.$

AND,

AND, after the same manner may any such common Surd quantity be reduced into an *Infinite Series*: But, with much greater ease and expedition,

99. ALL sorts of *fractional* and *surd* Quantities may be reduced into *Infinite Series* by the celebrated

BINOMIAL THEOREM *,

which is this, viz. $\overline{P+PQ}^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ$

$+ \frac{m-n}{2n} BQ + \frac{m-2n}{3n} CQ + \frac{m-3n}{4n} DQ +$

$\frac{m-4n}{5n} EQ + \&c.$ Wherein, $P+PQ$ repre-

sents the Quantity whose Power is to be thrown into an *Infinite Series*; P the First term of that quantity, which, in general, must be the Greatest; Q the Other, or Rest of the terms, divided by the First; $\frac{m}{n}$ the In-

dex of the power, whether it be affirmative or negative: and A= the First term of the *Series*, B= the Second, C= the Third, D = the Fourth, E= the Fifth, &c. that is,

A=

* The Truth of this Theorem has been demonstrated by various Writers; the Proof of it is therefore here omitted.

$$A = P^{\frac{m}{n}}, B = \frac{m}{n} AQ, C = \frac{m-n}{2n} BQ, D = \frac{m-2n}{3n} CQ, E = \frac{m-3n}{4n} DQ, \&c.$$

The following Examples will explain, and shew the great Use of this curious and noble Theorem.

EXAMPLE I.

100. To reduce $\sqrt{\frac{1}{2}a^2 + 4y^2}$ into an *Infinite Series*.

Here, $P = a^2$, $Q = \frac{4y^2}{a^2}$, $m = 1$, $n = 2$, $A = a$, $B = \frac{2y^2}{a}$, $C = -\frac{2y^4}{a^3}$, $D = \frac{4y^6}{a^5}$, $E = -\frac{10y^8}{a^7}$, &c. Therefore, $\sqrt{\frac{1}{2}a^2 + 4y^2}^{\frac{1}{2}} = a + \frac{2y^2}{a} - \frac{2y^4}{a^3} + \frac{4y^6}{a^5} - \frac{10y^8}{a^7}$, &c.

EXAMPLE II.

101. To reduce $\frac{1}{\sqrt{a^2 - y^2}}$, that is, $\sqrt{a^2 - y^2}^{-\frac{1}{2}}$, into an *Infinite Series*.

Here, $P = a^2$, $Q = -\frac{y^2}{a^2}$, $m = -1$, $n = 2$, $A = a^{-1} = \frac{1}{a}$, $B = \frac{y^2}{2a^3}$, $C = \frac{3y^4}{8a^5}$, $D = \frac{5y^6}{16a^7}$, $E =$

$$E = \frac{35y^8}{128a^9}, \&c. \text{ Therefore, } \overline{a^2 - y^2}^{-\frac{1}{2}} = \frac{1}{a} \\ + \frac{y^2}{2a^3} + \frac{3y^4}{8a^5} + \frac{5y^6}{16a^7} + \frac{35y^8}{128a^9}, \&c.$$

EXAMPLE III.

102. To reduce $\frac{1}{1-x}$, that is, $\overline{1-x}^{-1}$, into an *Infinite Series*.

Here, $P=1$, $Q=-x$, $m=-1$, $n=1$, $A=1$, $B=x$, $C=x^2$, $D=x^3$, $E=x^4$, &c. Therefore, $\overline{1-x}^{-1} = 1+x+x^2+x^3+x^4, \&c.$

EXAMPLE IV.

103. To reduce $\frac{1}{1+x}$, that is, $\overline{1+x}^{-1}$, into an *Infinite Series*.

Here, $P=1$, $Q=x$, $m=-1$, $n=1$, $A=1$, $B=-x$, $C=x^2$, $D=-x^3$, $E=x^4$, &c. Therefore, $\overline{1+x}^{-1} = 1-x+x^2-x^3+x^4, \&c.$

104. BUT, we may often find the *Series* answering to a proposed Quantity by the following

THEOREM,

$$\text{viz. } \overline{P+PQ}^{\frac{m}{n}} = P^{\frac{m}{n}} \times : 1 + \frac{m}{n} Q + \frac{m}{n} \times \frac{m-n}{2n} \\ \times Q^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} Q^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \\ \frac{m-2n}{3n}$$

$\frac{m-2n}{3^n} \times \frac{m-3n}{4^n} Q^* + \mathcal{E}c.$ (which, indeed, is the same as the former, though differently expressed,) with still greater ease and expedition; for, in this, no previous deduction is required; and both the numerators and denominators of the fractions $\frac{m}{n}, \frac{m-n}{2n}, \frac{m-2n}{3n}, \frac{m-3n}{4n}, \mathcal{E}c.$ are Series of numbers in arithmetical progression, which have the same common difference n . This will appear by the following Examples.—But, *note*, the Former Theorem is, in general, best adapted to shew the *Law* of the Series.

EXAMPLE I.

105. To reduce $\frac{1}{a^2+x^2}^{\frac{1}{2}}$, that is, $a^2+x^2)^{-\frac{1}{2}}$, into an *Infinite Series*.

Here, $P=a^2$, $Q=\frac{x^2}{a^2}$, $m=-1$, $n=2$.

Therefore, $a^2+x^2)^{-\frac{1}{2}} = \frac{1}{a} \times 1 - \frac{x^2}{2a^2} + \frac{3 \cdot x^4}{2 \cdot 4 \cdot a^4} - \frac{3 \cdot 5 \cdot x^6}{2 \cdot 4 \cdot 6 \cdot a^6} + \frac{3 \cdot 5 \cdot 7 \cdot x^8}{2 \cdot 4 \cdot 6 \cdot 8 \cdot a^8} - \mathcal{E}c. = \frac{1}{a} - \frac{x^2}{2a^2} + \frac{3x^4}{8a^4} - \frac{5x^6}{16a^6} + \frac{35x^8}{128a^8} - \mathcal{E}c.$

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Ex-

$$E = \frac{35y^3}{128a^9}, \&c. \text{ Therefore, } \overline{a^2 - y^2}^{-\frac{1}{2}} = \frac{1}{a} \\ + \frac{y^2}{2a^3} + \frac{3y^4}{8a^5} + \frac{5y^6}{16a^7} + \frac{35y^8}{128a^9}, \&c.$$

EXAMPLE III.

102. To reduce $\frac{1}{1-x}$, that is, $\overline{1-x}^{-1}$, into
to an *Infinite Series*.

Here, $P=1$, $Q=-x$, $m=-1$, $n=1$,
 $A=1$, $B=x$, $C=x^2$, $D=x^3$, $E=x^4$, &c.
Therefore, $\overline{1-x}^{-1} = 1+x+x^2+x^3+x^4, \&c.$

EXAMPLE IV.

103. To reduce $\frac{1}{1+x}$, that is, $\overline{1+x}^{-1}$, into
an *Infinite Series*.

Here, $P=1$, $Q=x$, $m=-1$, $n=1$, A
 $=1$, $B=-x$, $C=x^2$, $D=-x^3$, $E=x^4$, &c.
Therefore, $\overline{1+x}^{-1} = 1-x+x^2-x^3+x^4, \&c.$

104. BUT, we may often find the *Series*
answering to a proposed Quantity by the
following

THEOREM,

$$\text{viz. } \overline{P+PQ}^{\frac{m}{n}} = P^{\frac{m}{n}} \times : 1 + \frac{m}{n} Q + \frac{m}{n} \times \frac{m-n}{2n} \\ \times Q^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} Q^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \\ \frac{m-2n}{3n}$$

$\frac{m-2n}{3n} \times \frac{m-3n}{4n} Q^4 + \mathcal{E}c.$ (which, indeed, is the same as the former, though differently expressed,) with still greater ease and expedition; for, in this, no previous deduction is required; and both the numerators and denominators of the fractions $\frac{m}{n}, \frac{m-n}{2n}, \frac{m-2n}{3n}, \frac{m-3n}{4n}, \mathcal{E}c.$ are Series of numbers in arithmetical progression, which have the same common difference n . This will appear by the following Examples.—But, *note*, the Former Theorem is, in general, best adapted to shew the *Law* of the Series.

EXAMPLE I.

105. To reduce $\frac{1}{a^2+x^2}^{\frac{1}{2}}$, that is, $\overline{a^2+x^2}^{-\frac{1}{2}}$, into an *Infinite Series*.

Here, $P=a^2$, $Q=\frac{x^2}{a^2}$, $m=-1$, $n=2$.

Therefore, $\overline{a^2+x^2}^{-\frac{1}{2}} = \frac{1}{a} x : 1 - \frac{x^2}{2a^2} + \frac{3 \cdot x^4}{2 \cdot 4 \cdot a^4} - \frac{3 \cdot 5 \cdot x^6}{2 \cdot 4 \cdot 6 \cdot a^6} + \frac{3 \cdot 5 \cdot 7 \cdot x^8}{2 \cdot 4 \cdot 6 \cdot 8 \cdot a^8} - \mathcal{E}c. = \frac{1}{a} - \frac{x^2}{2a^2} + \frac{3x^4}{8a^4} - \frac{5x^6}{16a^6} + \frac{35x^8}{128a^8} - \mathcal{E}c.$

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E x-

EXAMPLE II.

106. To reduce $\overline{a+x}^{\frac{5}{3}}$ into an *Infinite Series*.

Here, $P=a$, $Q=\frac{x}{a}$, $m=5$, $n=3$. Therefore, $\overline{a+x}^{\frac{5}{3}} = a^{\frac{5}{3}} \times : 1 + \frac{5 \cdot x}{3 \cdot a} + \frac{5 \cdot 2 \cdot x^2}{3 \cdot 6 \cdot a^2} + \frac{5 \cdot 2 \cdot -1 \cdot x^3}{3 \cdot 6 \cdot 9 \cdot a^3} + \frac{5 \cdot 2 \cdot -1 \cdot -4 \cdot x^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot a^4} + \&c. = a^{\frac{5}{3}} + \frac{5a^{\frac{2}{3}}x}{3} + \frac{5x^2}{9a^{\frac{1}{3}}} - \frac{5x^3}{81a^{\frac{4}{3}}} + \frac{5x^4}{243a^{\frac{7}{3}}}, \&c.$

EXAMPLE III.

107. To reduce $\frac{b}{a+x}$, that is, $b \times \overline{a+x}^{-1}$, into an *Infinite Series*.

Here, $P=a$, $Q=\frac{x}{a}$, $m=-1$, $n=1$; and therefore, $\frac{m}{n}, \frac{m}{n} \times \frac{m-n}{2n}, \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}, \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n}, \&c.$ will be -1 and $+1$ alternately; and consequently $b \times \overline{a+x}^{-1}$ is $= b \times \frac{1}{a} \times : 1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3} + \frac{x^4}{a^4} - \&c. = \frac{b}{a} - \frac{bx}{a^2} + \frac{bx^2}{a^3} - \frac{bx^3}{a^4} + \frac{bx^4}{a^5} - \&c.$

E x;

EXAMPLE IV.

108. To reduce $\sqrt[4]{1-x}$ into an *Infinite Series*.

Here, $P=1$, $Q=-x$, $m=1$, $n=4$.

$$\begin{aligned} \text{Therefore, } \sqrt[4]{1-x} &= 1 - \frac{1}{4}x + \frac{1 \cdot -3}{4 \cdot 8} x^2 - \\ &\frac{1 \cdot -3 \cdot -7}{4 \cdot 8 \cdot 12} x^3 + \frac{1 \cdot -3 \cdot -7 \cdot -11}{4 \cdot 8 \cdot 12 \cdot 16} x^4 - \mathcal{E}c. \\ &= 1 - \frac{1}{4}x - \frac{3}{4 \cdot 8} x^2 - \frac{3 \cdot 7}{4 \cdot 8 \cdot 12} x^3 - \frac{3 \cdot 7 \cdot 11}{4 \cdot 8 \cdot 12 \cdot 16} x^4 \\ &- \mathcal{E}c, \end{aligned}$$

SCHOLIUM.

THE Sum of any Infinite geometrical series decreasing, is equal to the square of the first term divided by the difference between the first and second.

Thus, $a+x+\frac{x^2}{a}+\frac{x^3}{a^2}+\mathcal{E}c.$ is $=\frac{a^2}{a-x}$; and

$a-x+\frac{x^2}{a}-\frac{x^3}{a^2}+\mathcal{E}c.$ is $=\frac{a^2}{a+x}$. For, (art. 94.)

if a^2 be divided by $a-x$, and by $a+x$, the Quotients will be these infinite series of Terms decreasing in geometrical proportion continued.

C H A P. II.

Of finding the Fluent of a given Fluxion.

109. THE Business of the *Direct* Method of Fluxions being to find the Fluxion of a given Fluent, or the Velocity with which a Variable quantity flows at any point or term assigned; so the Business of the *Inverse* Method of Fluxions is to determine the Variable quantity, or Fluent, from that Velocity or Fluxion being given. And this, in General, may be done by the following *Rules*, these being the converse of those delivered in *Part 1. Chap. 2. **

R U L E I.

110. To find the Fluent of a Simple Fluxion, or, of that wherein there is no variable quantity and but One fluxional letter.

SUBSTITUTE the variable or flowing letter for it's Fluxion: and you will have the Fluent required.

Thus, the Fluent of ax is $= ax$. (*art. 14.*)

R U L E

* To treat at large on the different ways for finding the *Fluents* of the unbounded variety of Fluxional Expressions, would, by far, exceed the limits of an *introductory* Tract: This affair therefore cannot, with propriety, be handled here in so very extensive and copious a manner.

R U L E II.

111. To find the Fluent of a compound fluxional expression consisting of the products of two or more flowing quantities drawn into their Fluxions; that is, which consists of the Fluxion of each quantity drawn into the other or product of the rest of the quantities.

MULTIPLY the flowing quantities together: and the Product is the Fluent required.

Thus, the Fluent of $\dot{x}y + x\dot{y}$ is $= xy$; the Fluent of $\dot{x}yz + x\dot{y}z + xy\dot{z}$ is $= xyz$; and the Fluent of $\dot{v}xyz + v\dot{x}yz + vx\dot{y}z + vxy\dot{z}$ is $= vxyz$. (art. 15.)

R U L E III*.

112. To find the Fluent of a Fraction

$$\text{like } \frac{\dot{x}y - x\dot{y}}{y^2}.$$

DIVIDE the last term in the numerator by the Fluxion of the Negative square root of the denominator; then, divide this quotient by the Affirmative square root of the denominator:

* This Rule must be used with caution, as it is not applicable to fractional expressions in general.

minator: and you will have the Fluent required.

Thus, the Fluent of $\frac{\dot{x}y - x\dot{y}}{y^2}$ is $= \frac{x}{y}$. (art. 17.)

R U L E IV.

113. To find the Fluent of an expression compounded of different fluxionary terms connected together by the Signs + and —.

FIND the separate Fluents of the different terms; which connect together by the Signs of their respective Fluxions: and you will have the Fluent required.

Thus, the Fluent of $a\dot{x} + \dot{x}y + x\dot{y} - \frac{\dot{x}y - x\dot{y}}{y^2}$ is $= ax + xy - \frac{x}{y}$. (art. 19.)

R U L E V.

114. To find the Fluent of an expression which consists of the Fluxion of a variable quantity drawn into any Power of that quantity contained any number of times*.

1^o. In Simple Expressions, — Strike out the *fluxional* letter; add 1 to the Index of the

* The Rule fails when the Index of the Power is —1. To find the Fluent then, see Rule 6.

the Power; and divide by the Index thus increased: and you will have the Fluent required.

Thus, the Fluent of $2x^2\dot{x}$ is $= \frac{2}{3}x^3$; the Fluent of $-\frac{\dot{x}}{x^2}$, that is, of $-x^{-2}\dot{x}$, is $= -x^{-2+1}$ divided by $-2+1$, that is, $= \frac{-x^{-1}}{-1} = x^{-1} = \frac{1}{x}$. And, Universally, the Fluent of $mx^{m-1}\dot{x}$ is $= x^m$; and the Fluent of $\frac{m}{n}x^{m-n}\dot{x}$ is $= x^{\frac{m}{n}}$.

2°. In Compound Expressions,—where the *fluxionary* part is equal, or in an invariable ratio, to the Fluxion of the quantity under the vinculum,—Add 1 to the Index of the Power; and divide by the Fluxion of the quantity under the vinculum, drawn into the Index of the Power thus increased: and you will have the Fluent required.

Thus, the Fluent of $\overline{x+x^2}^2 \times 3\dot{x} + 6x\dot{x}$ is $= \frac{\overline{x+x^2}^3 \times 3\dot{x} + 6x\dot{x}}{3 \times \dot{x} + 2x\dot{x}} = \overline{x+x^2}^3$; and

the Fluent of $-\frac{2}{3} \times \overline{x-a}^{-\frac{5}{3}} \times \dot{x}$ is $= \frac{-\frac{2}{3} \times \overline{x-a}^{-\frac{5}{3}+1} \times \dot{x}}{-\frac{5}{3}+1 \times \dot{x}} = \frac{-\frac{2}{3} \times \overline{x-a}^{-\frac{2}{3}} \times \dot{x}}{-\frac{2}{3}\dot{x}} = \overline{x-a}^{-\frac{2}{3}}$. (art. 20.)

RULE

RULE VI.

115. To find the Fluent of a compound fluxional expression, like $\frac{b}{a+x} \dot{x}$, or

$$\frac{b\dot{x}}{a^2+x^2}^{\frac{1}{2}}; \&c.$$

1°. THROW the expression into an Infinite Series; and find the Fluent of the Series by the foregoing Rules: and you will have the Fluent required.

Thus, to find the Fluent of $\frac{b}{a+x} \dot{x}$; throw the expression into a Series; which (*art. 94.*) is $= \frac{b\dot{x}}{a} - \frac{bx\dot{x}}{a^2} + \frac{bx^2\dot{x}}{a^3} - \frac{bx^3\dot{x}}{a^4} + \&c.$ and then find the Fluent of this Series; which, by the preceding Rules, is $= \frac{bx}{a} - \frac{bx^2}{2a^2} + \frac{bx^3}{3a^3} - \frac{bx^4}{4a^4} + \&c.$ and is the Fluent required.

And, to find the Fluent of $\frac{b\dot{x}}{a^2+x^2}^{\frac{1}{2}}$; throw the expression into a Series; which, (*art. 105.*) is $= b\dot{x} \times : \frac{1}{a} - \frac{x^2}{2a^3} + \frac{3x^4}{8a^5} - \frac{5x^6}{16a^7} + \&c. =$
 $\frac{b\dot{x}}{a} - \frac{bx^2\dot{x}}{2a^3} + \frac{3bx^4\dot{x}}{8a^5} - \frac{5bx^6\dot{x}}{16a^7} + \&c.$ Now, the
 Fluent

Fluent of this Series, by the foregoing Rules, is $\frac{bx}{a} - \frac{bx^3}{6a^3} + \frac{3bx^5}{40a^5} - \frac{5bx^7}{112a^7} + \&c.$ which is the Fluent required.

2^o. OR, Because (*art.* 21.) the Fluxion of the Hyperbolic Logarithm of any quantity is equal to the Fluxion of that quantity, divided by the quantity itself; therefore,

The Fluent of $b \times \frac{\dot{x}}{a+x}$ is $= b \times$ Hyp. Log. of $a+x$. For, the Fluxion of $a+x$ is $\dot{x}$, which divided by $a+x$ is $\frac{\dot{x}}{a+x}$.

And the Fluent of $b \times \frac{\dot{x}}{a^2+x^2}^{\frac{1}{2}}$ is $= b \times$

Hyp. Log. of $x + \sqrt{a^2+x^2}^{\frac{1}{2}}$. For, the Fluxion of $x + \sqrt{a^2+x^2}^{\frac{1}{2}}$ is $\dot{x} + \frac{x\dot{x}}{a^2+x^2}^{\frac{1}{2}} =$

$$\frac{\dot{x} \times \sqrt{a^2+x^2}^{\frac{1}{2}} + x\dot{x}}{a^2+x^2}^{\frac{1}{2}} = \frac{\dot{x}}{a^2+x^2}^{\frac{1}{2}} \times x + \sqrt{a^2+x^2}^{\frac{1}{2}},$$

which divided by $x + \sqrt{a^2+x^2}^{\frac{1}{2}}$ is $\frac{\dot{x}}{a^2+x^2}^{\frac{1}{2}}$.

&c.

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116. **THOUGH** the Fluxion of any Fluent, how much soever compounded it be, may be accurately found; yet, the Fluents of compound Fluxional expressions cannot always be had in finite terms.

117. **THOUGH** no Fluent can have more than One Fluxion; yet, a Fluxion may have an Infinite number of Fluents. Thus, for Example, the Fluent of $\dot{x}$ may be either x or $x \pm a$; wherein, a represents any invariable quantity whatsoever.—Now, to find a , when it must be added to or taken from the Fluent x , is called *correcting* the Fluent: And to effect this; in any Equation, after having obtained the Fluent of each side by the foregoing Rules; make the variable letter in either of them vanish, or equal to nothing; and substitute for the variable letter in the other, such a determinate or invariable value as it is then known to have: or, for the variable quantities, write such invariable or fixed values as they are respectively known to have at any particular point or term. Then, if we subtract the sides of this new Equation from the corresponding Fluents before found; the remaining Fluents will
be

be Contemporary, or always equal to each other; and consequently, we shall then have the correct Fluent sought determined.— This affair may, perhaps, be better understood by giving the following Examples.

Example 1.

To find the Correct Fluent of $\dot{y} = 2x\dot{x}$.

The Fluent of this equation, by *art.* 114. is $y = x^2$. Now, when $y = 0$, if $x = 0$; then, $y - 0 = x^2 - 0$, or $y = x^2$: therefore, the Fluent first found needs no correction.

Example 2.

To find the Correct Fluent of $a\dot{x} - 2x\dot{y} = 2y\dot{y}$.

The Fluent of this equation, by *art.* 114. is $ax - x^2 = y^2$. Now, when y ends, or when $y = 0$, if x be $= a$; then, substituting 0 for y , and a for x , the fluential equation will become $a^2 - a^2 = 0$, that is $0 = 0$: therefore, the Fluent first found needs no correction.

Example 3.

To find the Correct Fluent of $\dot{z} = a\dot{y}$.

The Fluent of this equation, by *art.* 110. is $z = ay$. Now, if y be $= b$ when z is $= 0$;

X 2

then,

156 *An INTRODUCTION to the*

then, by writing in this fluential equation a for z and b for y , it will become $0 = ab$; therefore, ay is always ab greater than z ; and consequently, the Fluent corrected is $z = ay - ab$.

Example 4.

To find the Correct Fluent of $y = -\frac{1}{(a+x)^3} \times 2\dot{x}$.

The Fluent of this equation, by *art. 114*.

$$\text{is } y = \frac{-\frac{1}{(a+x)^3} \times 2\dot{x}}{-2\dot{x}} = \frac{1}{(a+x)^3} = \frac{1}{(a+x)^2}.$$

Now, if $y=0$ when $x=0$; then, this fluential equation will become $0 = \frac{1}{a^2}$: and

therefore, y is always less than $\frac{1}{(a+x)^2}$ by the

quantity $\frac{1}{a^2}$: consequently, the Contempo-

rary Fluents will be $y = \frac{1}{(a+x)^2} - \frac{1}{a^2}$.

Example 5.

To find the Correct Fluent of $\dot{z} =$

$$b \times \frac{\dot{x}}{(a^2 + x^2)^{\frac{1}{2}}}$$

The Fluxion of the Hyp. Log. of $x + \sqrt{a^2 + x^2}$ is $= \frac{\dot{x}}{(a^2 + x^2)^{\frac{1}{2}}}$ (*art. 21. or 115.*)

there-

therefore, the Fluent of $\dot{z} = b \times \frac{\dot{x}}{a^2 + x^2}^{\frac{1}{2}}$ is

$z = b \times \text{Hyp. Log. of } x + \sqrt{a^2 + x^2}^{\frac{1}{2}}$. Now, when $z = 0$, if x be likewise $= 0$, this Fluent will then become $0 = b \times \text{Hyp. Log. of } a$; which subtracted from the said Fluent, makes the Correct Fluent, or *true* value of $z = b \times \text{Hyp. Log. of } x + \sqrt{a^2 + x^2}^{\frac{1}{2}} - b \times \text{Hyp. Log. of } a$ (which, by the Nature of Logarithms, is) $= b \times \text{Hyp. Log. of } \frac{x + \sqrt{a^2 + x^2}^{\frac{1}{2}}}{a}$.

Example 6.

To find the Correct Fluent of $a\dot{x} - bx\dot{x} = y\dot{y}$.

The Fluent of this equation (*art.* 114.) is $ax - \frac{1}{2}bx^2 = \frac{1}{2}y^2$. Now, if $x = c$ when $y = d$; then, substituting c for x and d for y , the equation will be $ac - \frac{1}{2}bc^2 = \frac{1}{2}d^2$; and therefore, by subtracting the corresponding sides of this equation from the above, we shall have the Correct or Contemporary Fluents $ax - \frac{1}{2}bx^2 - ac + \frac{1}{2}bc^2 = \frac{1}{2}y^2 - \frac{1}{2}d^2$.

C H A P. III.

Of finding the Length of a Curve Line.

Fig. 118. IN Curves referred to an Axis, (*fig.* 57.) let cb be supposed indefinitely near and parallel to the ordinate CB , and Bn equal and parallel to Cc the Increment of the absciss AC : And in Curves referred to a fixed or central Point, (*fig.* 58.) let bC be supposed indefinitely near to BC , and the indefinitely little circular arch Bn be described with the ordinate or radius CB . Put AC (*fig.* 57.) $=x$, $CB=y$, curve $AB=z$; $Bn=x'$, $nb=y'$, and $Bb=z'$: then, (the Increment Bb being considered as a little right line,) by 47 E. 1. $Bb=\overline{Bn^2+nb^2}^{\frac{1}{2}}$, that is, $z'=\overline{x'^2+y'^2}^{\frac{1}{2}}$, or (*art.* 7.) $\dot{z}=\overline{\dot{x}^2+\dot{y}^2}^{\frac{1}{2}}$. Put the ordinate CB (*fig.* 58.) $=y$, curve $CPB=z$, $Bn=x'$, $nb=y'$, and $Bb=z'$: then, (because Bb may be considered as an indefinitely small right line, and Bn as a little right line perpendicular to Cb ,) as before, $z'=\overline{x'^2+y'^2}^{\frac{1}{2}}$, or $\dot{z}=\overline{\dot{x}^2+\dot{y}^2}^{\frac{1}{2}}$. And this is a General Expression for the Fluxion of the Length of any Curve Line whatsoever *.

Now,

* The same expression may be derived, without the help of *Increments*, from *art.* 24 and 38.

Now, by help of the Equation or Properties of the given Curve whose Length is required, we may find the value of $\dot{x}^2$ in terms of $\dot{y}^2$, or of $\dot{y}^2$ in terms of $\dot{x}^2$; and then, by substitution, $\dot{x}^2$ or $\dot{y}^2$ in this General Expression will be exterminated: and by finding the Fluent of the resulting equation, we shall have the value of z , or the Length of the Curve required.

EXAMPLE I.

119. To find the Length of the Curve AB *Fig.*
 $=z$, whose Equation (putting the given *59.*
 line $AG = \frac{2}{3}a$, $GC = x$, and $CB = y$), is
 $2x\sqrt{a^2 + x^2}^{\frac{3}{2}} = 3a^2y$.

The Fluxion of this equation is $3x\sqrt{a^2 + x^2}^{\frac{3}{2}} \times 2x\dot{x} = 3a^2\dot{y}$; therefore, $\dot{y} = \frac{2x\dot{x}}{3a^2} \times 3x\sqrt{a^2 + x^2}^{\frac{1}{2}}$
 $= \frac{2x\dot{x}}{a^2} \times \sqrt{a^2 + x^2}^{\frac{1}{2}}$, and $\dot{y}^2 = \frac{4x^2\dot{x}^2}{a^4} \sqrt{a^2 + x^2}$
 $= \frac{4a^2x^2\dot{x}^2 + 4x^4\dot{x}^2}{a^4}$; which substituted
 for $\dot{y}^2$, makes $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}^{\frac{1}{2}}$ (*art.* 118.) =
 $\sqrt{\frac{a^4\dot{x}^2 + 4a^2x^2\dot{x}^2 + 4x^4\dot{x}^2}{a^4}}^{\frac{1}{2}} = \frac{a^2\dot{x} + 2x^2\dot{x}}{a^2} =$ the
 Fluxion of the Curve AB; whose Fluent is
 $z =$

$z = \frac{a^2 x + \frac{2}{3} x^3}{a^2} = x + \frac{2x^3}{3a^2} =$ the Length of the Curve AB required.

EXAMPLE II.

120. To find the Length of the common Cycloid*.

Put OA or OF the radius of the generating circle $= a$, absciss AC $= x$, ordinate CB $= y$, CG $= s$, and arch AB $= z$; then, (as was found in *art.* 35.) $\dot{y} = \frac{2a-x}{s} \dot{x}$, and

therefore $\dot{y}^2 = \frac{(2a-x)^2}{s^2} \dot{x}^2$; which substituted for $\dot{y}^2$, makes the general expression for the Fluxion of the Curve (*art.* 118.)

$$\begin{aligned} \dot{z} &= \left(\dot{x}^2 + \dot{y}^2 \right)^{\frac{1}{2}} = \left(\dot{x}^2 + \frac{(2a-x)^2}{s^2} \dot{x}^2 \right)^{\frac{1}{2}} = \\ &= \left(\frac{s^2 \dot{x}^2 + 4a^2 \dot{x}^2 - 4ax \dot{x}^2 + x^2 \dot{x}^2}{s^2} \right)^{\frac{1}{2}}, \text{ that is, (because by 35 E. 3. } GC^2 = AC \times CF, \text{ or } s^2 = \\ &2ax - x^2,) } \dot{z} = \left(\frac{4a^2 \dot{x}^2 - 2ax \dot{x}^2}{2ax - x^2} \right)^{\frac{1}{2}} = \left(\frac{2a \dot{x}^2}{x} \right)^{\frac{1}{2}} = \\ &2a^{\frac{1}{2}} \times x^{-\frac{1}{2}} \dot{x}; \text{ and the Fluent of this is} \\ & \qquad \qquad \qquad z = \end{aligned}$$

* See the generation of this Curve, *art.* 35. *note.*

$\dot{x} = \sqrt{2a}^{\frac{1}{2}} \times 2\dot{x}^{\frac{1}{2}} = 2 \times \sqrt{2ax}^{\frac{1}{2}} =$ twice the chord AG; (for, the triangles FAG and GAC being similar, FA:AG::GA:AC, or $\sqrt{FA \times AC}^{\frac{1}{2}} = AG$, that is, $\sqrt{2ax}^{\frac{1}{2}} = AG$.) Whence, by writing $2a$ for x , the Length of the Semicycloid AD appears to be equal to twice the diameter AF of it's generating Semicircle.

EXAMPLE III.

121. To find the Length of a *Parabola*.

Put the parameter $= a$, absciss AC $= x$, ordinate CB $= y$, and curve AB $= z$; then Fig. 61.

(art. 28.) $\dot{x} = \frac{2y\dot{y}}{a}$, and therefore $\dot{x}^2 = \frac{4y^2\dot{y}^2}{a^2}$;

which substituted for $\dot{x}^2$, makes $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}^{\frac{1}{2}}$ (the general expression for the Fluxion of the

Length of the Curve, art. 118.) $= \sqrt{\frac{4y^2\dot{y}^2}{a^2} + \dot{y}^2}^{\frac{1}{2}}$

$= \frac{\dot{y}}{a} \times \sqrt{a^2 + 4y^2}^{\frac{1}{2}}$; which, thrown into an In-

finite Series, (art. 100.) is $= \frac{\dot{y}}{a} \times \left(a + \frac{2y^2}{a} - \right.$

$\frac{2y^4}{a^3} + \frac{4y^6}{a^5} - \&c.$ that is, $\dot{z} = \dot{y} + \frac{2y^2\dot{y}}{a^2} - \frac{2y^4\dot{y}}{a^4}$

$+ \frac{4y^6\dot{y}}{a^6} - \&c.$ And the Fluent of this Series

is $z = y + \frac{2y^3}{3a^2} - \frac{2y^5}{5a^4} + \frac{4y^7}{7a^6} - \text{Ec.} =$ the Length of the Curve AB required.

Or, The above $\dot{z} = \frac{\dot{y}}{a} \times \overline{a^2 + 4y^2}^{\frac{1}{2}}$ is =
 $\frac{\dot{y} \times \overline{a^2 + 4y^2}}{a \times \overline{a^2 + 4y^2}^{\frac{1}{2}}} = \frac{a^2 y \dot{y} + 4y^3 \dot{y}}{a \times \overline{a^2 y^2 + 4y^4}^{\frac{1}{2}}} = \frac{\frac{1}{2} a^2 y \dot{y} + 4y^3 \dot{y}}{a \times \overline{a^2 y^2 + 4y^4}^{\frac{1}{2}}}$
 $+ \frac{\frac{1}{2} a^2 \dot{y}}{a \times \overline{a^2 + 4y^2}^{\frac{1}{2}}} = \frac{1}{a} \times \overline{a^2 y^2 + 4y^4}^{-\frac{1}{2}} \times$
 $\frac{\frac{1}{2} a^2 y \dot{y} + 4y^3 \dot{y}}{\frac{1}{4} a^2 + y^2}^{\frac{1}{2}} + \frac{1}{4} a \times \frac{\dot{y}}{\overline{\frac{1}{4} a^2 + y^2}^{\frac{1}{2}}}$. Now, (*art.*
 114.) the Fluent of the first of these two
 terms is $= \frac{1}{2a} \times \overline{a^2 y^2 + 4y^4}^{\frac{1}{2}} = \frac{y}{a} \times \overline{\frac{1}{4} a^2 + y^2}^{\frac{1}{2}}$;
 and the Fluent of the last of the said two
 terms (*art.* 115.) is $= \frac{1}{4} a \times$ Hyp. Log. of
 $y + \overline{\frac{1}{4} a^2 + y^2}^{\frac{1}{2}}$; therefore, $z = \frac{y}{a} \times \overline{\frac{1}{4} a^2 + y^2}^{\frac{1}{2}}$
 $+ \frac{1}{4} a \times$ Hyp. Log. of $y + \overline{\frac{1}{4} a^2 + y^2}^{\frac{1}{2}}$. But,
 since when z and y vanish or become $= 0$,
 (as at the vertex A,) this Fluent becomes $=$
 $\frac{1}{4} a \times$ Hyp. Log. of $\frac{1}{4} a$; therefore the said
 Fluent being Corrected (*art.* 117. *Ex.* 5.)
 makes the *true* value of z or the Length of
 the Curve AB $= \frac{y}{a} \times \overline{\frac{1}{4} a^2 + y^2}^{\frac{1}{2}} + \frac{1}{4} a \times$ Hyp.
 Log.

Log. of $y + \sqrt{\frac{1}{4}a^2 + y^2}$ $^{\frac{1}{2}}$ — $\frac{1}{4}a \times$ Hyp. Log. of $\frac{1}{2}a = \frac{y}{a} \times \sqrt{\frac{1}{4}a^2 + y^2}$ $^{\frac{1}{2}}$ + $\frac{1}{4}a \times$ Hyp. Log. of $\frac{y + \sqrt{\frac{1}{4}a^2 + y^2}}{\frac{1}{2}a}$ $^{\frac{1}{2}}$, by the Nature of Logarithms.

EXAMPLE IV.

122. To find the Length of any Arch of a Circle.

Put the radius $EA = a$, absciss $AC = x$, ordinate $CB = y$, arch $AB = z$: then, (art. *Fig.* 62.)

$$27.) \dot{x} = \frac{y\dot{y}}{a-x} = (\text{because } a-x = CE =$$

$$\sqrt{EA^2 - BC^2} = \sqrt{a^2 - y^2}) \frac{y\dot{y}}{\sqrt{a^2 - y^2}}; \text{ there-}$$

fore $\dot{x}^2 = \frac{y^2 \dot{y}^2}{a^2 - y^2}$; which substituted for $\dot{x}^2$,

makes $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$ (the general expression for the Fluxion of the Length of the Curve,

$$\text{art. 118.}) = \sqrt{\frac{y^2 \dot{y}^2}{a^2 - y^2} + \dot{y}^2} = \frac{a\dot{y}}{\sqrt{a^2 - y^2}} = a\dot{y} \times$$

$\sqrt{a^2 - y^2}^{-\frac{1}{2}}$; which thrown into an Infinite

Series (art. 101.) is $\dot{z} = a\dot{y} \times \frac{1}{a} + \frac{y^2}{2a^3} +$

$$\frac{3y^4}{8a^3} + \frac{5y^6}{16a^7} + \frac{35y^8}{128a^9} + \frac{63y^{10}}{256a^{11}} + \frac{231y^{12}}{1024a^{13}} + \frac{429y^{14}}{2048a^{15}} + \mathcal{E}c. = y + \frac{y^2 \dot{y}}{2a^2} + \frac{3y^4 \dot{y}}{8a^4} + \frac{5y^6 \dot{y}}{16a^6} + \frac{35y^8 \dot{y}}{128a^8} + \frac{63y^{10} \dot{y}}{256a^{10}} + \frac{231y^{12} \dot{y}}{1024a^{12}} + \frac{429y^{14} \dot{y}}{2048a^{14}} + \mathcal{E}c.$$

And the Fluent of this Series (*art.* 114.) is

$$x = y + \frac{y^3}{6a^2} + \frac{3y^5}{40a^4} + \frac{5y^7}{112a^6} + \frac{35y^9}{1152a^8} + \frac{63y^{11}}{2816a^{10}} + \frac{231y^{13}}{13312a^{12}} + \frac{143y^{15}}{10240a^{14}} + \mathcal{E}c. =$$

the Length of the Arch AB.

Now, if we suppose the radius $EA = a = 1$, and the $\angle AEB = 30^\circ$. then, (because the sine of any arch is equal to half the chord of twice that arch, and the chord of 60° . is equal to the radius,) the sine, or ordinate $CB = y$, will be $= \frac{1}{2}$; and therefore, the terms of the above Fluent being reduced into Decimal Fractions, and placed under one another, will stand thus; viz.

$$+ \frac{1}{6} + \frac{1}{40} + \frac{1}{112} + \frac{1}{1152} + \frac{1}{2816} + \frac{1}{13312} + \frac{1}{10240} + \dots = 0.500000000$$

DOCTRINE of FLUXIONS. 165

.5000000000 &c.
 .020833333
 .002343750
 .000348772
 .000059339
 .000010923
 .000002118
 .000000426
 &c.

The sum } .5235987 &c. = { the Length of
 of which is }
 which x by 12

is 6.283185 &c. = { the Periphery of
 a Circle whose
 Radius is 1;
 therefore 3.141592 &c. = { the Periphery of
 a Circle whose
 Diameter is 1.

EXAMPLE V.

123. To find the Length of any Arch of
Archimedes's Spiral *.

Put CA the radius of the generating Fig.
 circle = b , and ARA the circumference of 63.
 it = a ; also, put the ordinate CB = y , the
 length of the required arch CPB = z , and a
 cir=

* See how this Curve is generated, art. 39. note.

circular arch whose radius is the ordinate $CB=x$. Then, (as was found in *art.* 39.)

$$\dot{x} = \frac{ay\dot{y}}{b^2}, \text{ and therefore } \dot{x}^2 = \frac{a^2y^2\dot{y}^2}{b^4}; \text{ which}$$

substituted for $\dot{x}^2$, makes the general expression for the Fluxion of the Curve, viz.

$$\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}^{\frac{1}{2}} \text{ (art. 118.)} = \sqrt{\frac{a^2y^2\dot{y}^2}{b^4} + \dot{y}^2}^{\frac{1}{2}} =$$

$$\frac{\dot{y}}{b^2} \times \sqrt{a^2y^2 + b^4}^{\frac{1}{2}} = \frac{\dot{y} \times a^2y^2 + b^4}{b^2 \times \sqrt{a^2y^2 + b^4}^{\frac{1}{2}}} =$$

$$\frac{a^2y^3\dot{y} + b^4\dot{y}}{b^2 \times \sqrt{a^2y^2 + b^4}^{\frac{1}{2}}} = \frac{a^2y^3\dot{y} + \frac{1}{2}b^4\dot{y}}{b^2 \times \sqrt{a^2y^2 + b^4}^{\frac{1}{2}}} +$$

$$\frac{\frac{1}{2}b^4\dot{y}}{b^2 \times \sqrt{a^2y^2 + b^4}^{\frac{1}{2}}} = \frac{1}{b^2} \times \sqrt{a^2y^2 + b^4}^{-\frac{1}{2}} \times$$

$$\frac{a^2y^3\dot{y} + \frac{1}{2}b^4\dot{y}}{a^2y^2 + b^4}^{\frac{1}{2}} + \frac{b^2}{2a} \times \frac{a\dot{y}}{\sqrt{a^2y^2 + b^4}^{\frac{1}{2}}}. \text{ Now, the}$$

Fluent of the first of these two terms is

$$\text{found by art. 114.} = \frac{1}{2b^2} \times \sqrt{a^2y^2 + b^4}^{\frac{1}{2}}; \text{ and}$$

the Fluent of the last is found by *art.* 115.

$$= \frac{b^2}{2a} \times \text{Hyp. Log. of } ay + \sqrt{a^2y^2 + b^4}^{\frac{1}{2}}; \text{ there-}$$

$$\text{fore } z = \frac{1}{2b^2} \times \sqrt{a^2y^2 + b^4}^{\frac{1}{2}} + \frac{b^2}{2a} \times \text{Hyp. Log.}$$

$$\text{of } ay + \sqrt{a^2y^2 + b^4}^{\frac{1}{2}}: \text{ but, when } z \text{ and } y \text{ are}$$

$$= 0,$$

$= 0$, as at the point C, this equation becomes $0 = \frac{b^2}{2a} \times \text{Hyp. Log. of } b^2$; and therefore, the Fluent Corrected, (*art.* 117.) makes

the true value of $z = \frac{1}{2b^2} \times \sqrt{a^2y^4 + b^4y^2}^{\frac{1}{2}} +$

$\frac{b^2}{2a} \times \text{Hyp. Log. of } ay + \sqrt{a^2y^2 + b^4}^{\frac{1}{2}} - \frac{b^2}{2a} \times$

$\text{Hyp. Log. of } b^2 =$ (by the Nature of Logarithms; the Difference of the Logarithms of any two Numbers being equal to the Logarithm of their Quotient;) $\frac{1}{2b^2} \times \sqrt{a^2y^4 + b^4y^2}^{\frac{1}{2}} +$

$+\frac{b^2}{2a} \times \text{Hyp. Log. of } \frac{ay + \sqrt{a^2y^2 + b^4}^{\frac{1}{2}}}{b^2} =$ the

Length of the Arch CPB required. And therefore, by substituting the radius b for the ordinate y , we shall have the Length of

the whole Spiral CPBA $= \frac{1}{2} \times \sqrt{a^2 + b^2}^{\frac{1}{2}} + \frac{b^2}{2a}$

$\times \text{Hyp. Log. of } \frac{a + \sqrt{a^2 + b^2}^{\frac{1}{2}}}{b}$.

C H A P. IV.

Of finding the Areas of Curvilinear Spaces.

Fig. 124. IN Curves whose ordinates are referred to an Axis (*fig.* 64.) let bc be conceived indefinitely near and parallel to the perpendicular ordinate BC , and Bn equal and parallel to Cc the Increment of the absciss AC : then, because bn bears no assignable ratio to BC , bc may be taken as equal to BC or nc ; and the trapezium $BCcb$ as equal to the parallelogram $BCcn$: but, $BCcb$ is the Moment or Increment of the curvilinear space ABC , that is, (if we put $AC=x$, $CB=y$, and $Cc=x'$,) the Moment or Increment of the Space ABC is $= yx'$; and therefore, (*art.* 7.) the Fluxion of it is $= y\dot{x}$.—In like manner, in Spirals, or those Curves whose ordinates are referred to a fixed or central Point (*fig.* 65.) let bC be conceived indefinitely near to the ordinate BC , and the little circular arch Bn (whose radius is CB ,) be supposed a little right line perpendicular to bC : then, bn having less than any assignable ratio to nC , BCn may be considered as equal to BCb the Moment or Increment of the curvilinear Space $BPCB$; that is, (if we put $CB=y$, and $Bn=x'$,) the
Moment

Moment or Increment of the Space CPBC is $= \frac{1}{2}yx'$, or it's Fluxion $= \frac{1}{2}y\dot{x}$.

Or, Let the curvilinear space AEI and parallelogram AG (*fig. 64.*) be generated by the perpendicular and indefinite right line AF moving with a parallel motion from A along the axis AE; then, it is evident, the curvilinear space will increase slower, or flow with a less degree of velocity, than the parallelogram, before the said generating line arrives at the term CB; and afterwards faster, or with a greater degree of velocity: therefore, at the said term, they will flow with one and the same degree of velocity; that is, at the term CB, the Fluxions of the curvilinear space and parallelogram will be equal: But, it is plain, the Fluxion of the parallelogram, at the term CB, is equal to DA or BC drawn into the Fluxion of AC; therefore, the Fluxion of the curvilinear space ACB is equal to the ordinate CB drawn into the Fluxion of the absciss AC; that is, (putting $AC=x$, and $CB=y$,) the Fluxion of the curvilinear Space ACB is $= y\dot{x}$.—In *fig. 65.* let the curvilinear space CPIC be generated by the variable right line CF turning round the center C; and, at the same time, let the sector CDGC be described

Z

by

by the radius CD; then, it is plain, before the line CF comes to be in the situation CB, the spiral space will increase slower, or flow with a less degree of velocity, than the circular space or sector CDG; and afterwards faster, or with a greater degree of velocity: therefore, at the term CB, they will increase or flow with an equal degree of velocity: But, it is evident, the velocity with which the sector enlarges, is equal to half it's radius drawn into the velocity with which it's arch is described; therefore, the velocity with which the curvilinear space CPBC is increased, at the term CB, is equal to $\frac{1}{2}$ CB drawn into the velocity of the point D or B moving along the arch DG at the point B; that is, the Fluxion of the said curvilinear space is equal to $\frac{1}{2}$ CB drawn into the Fluxion of the circular arch DB; or (putting the ordinate CB= y , and arch DB= x ,) the Fluxion of the curvilinear Space CPBC is $=\frac{1}{2}y\dot{x}$; as before.

125. WHEREFORE, when the Curve is referred to an Axis, (*fig. 64.*) find the value of y in terms of x , by help of the Equation of the given Curve, which multiply by $\dot{x}$; or, find the value of $\dot{x}$ in terms of y , which multiply by y : Then, the Fluent of the resulting

sulting fluxional expression will express the Area (or quadrature) of the curvilinear Space ABC required.—And, when the Curve is referred to a fixed or central Point C, (*fig. 65.*) find the value of $\dot{x}$ in terms of $\dot{y}$, from the properties of the given curve; then multiply this value of $\dot{x}$ by $\frac{1}{2}y$, and find the Fluent; and it will give the Area of the Space required.

EXAMPLE I.

126. To find the Area of the Space ABCA; *Fig. 66.*
the Curve AB being a *Parabola*.

Put the absciss $AC=x$, ordinate $CB=y$, and the parameter $=1$. Then, by the nature of the curve, $x=y^2$, or $x^{\frac{1}{2}}=y$; therefore, $y\dot{x}$ (the Fluxion of the Area, *art. 124.*) $=x^{\frac{1}{2}}\dot{x}$; the Fluent of which is $\frac{2}{3}x^{\frac{3}{2}}=$ (by substituting y for $x^{\frac{1}{2}}$ it's value,) $\frac{2}{3}xy =$ the Area required.

Or, The Fluxion of the above equation of the curve, viz. of $x=y^2$, is $\dot{x}=2y\dot{y}$; which, multiplied by y , makes the Fluxion of the Area, viz. $y\dot{x}$ (*art. 124.*) $=2y^2\dot{y}$; the Fluent of which is $\frac{2}{3}y^3=$ (by writing x for y^2 it's value,) $\frac{2}{3}xy$; as before.

Corollary.

The Area of any Parabolic Space ABCA is equal to two-third-parts of it's circumscribing Parallelogram ADBC.

EXAMPLE II.

Fig. 127. To find the Area of the Space ABCG; 67. the property of the Curve AB being such, that it's Subtangent CT is invariable, or always of the same value. (See *art.* 80.)

Put the given Subtangent $CT=a$, $GA=b$, $GC=x$, and $CB=y$. Then, (*art.* 25.) $a=\frac{\dot{x}y}{\dot{y}}$; therefore, $\dot{x}=\frac{a\dot{y}}{y}$; which, multiplied by y , makes the Fluxion of the Space viz. $y\dot{x}$ (*art.* 124.) $=a\dot{y}$; and the Fluent of this Fluxion is ay : But, when the area of the space is $=0$, or $y=b$, this expression for the Fluent becomes $=ab$; and therefore, (*art.* 117. *Ex.* 3.) the Fluent corrected is $ay-ab = y-bx = a$ the Area of the Space ABCG required.

EXAMPLE III.

Fig. 128. To find the Area of the Space CPBC; 63. the Curve CPBA being the *Spiral of Archimedes* *.

Put the circumference of the generating circle $ARA=a$, and it's radius $CA=b$; also,

* See the generation of this Curve, *art.* 39. *note.*

DOCTRINE of FLUXIONS. 173

also, put the ordinate $CB=y$. Now, (as was found in *art.* 39.) $\dot{x}=\frac{ay\dot{y}}{b^2}$; which equation multiplied by $\frac{1}{2}y$ makes the Fluxion of the Space viz. $\frac{1}{2}y\dot{x}$ (*art.* 124.) $=\frac{ay\dot{y}}{b^2} \times \frac{1}{2}y = \frac{ay^2\dot{y}}{2b^2}$; the Fluent of which is $\frac{ay^3}{6b^2}$ = the Area of the Space required. And therefore, by substituting b for y , we have the Area of the whole spiral Space $CPBAC = \frac{1}{6}ab$; which, because the area of a circle is equal to it's periphery drawn into half the radius, is $=\frac{1}{3}$ of the Area of the generating Circle.

EXAMPLE IV.

129. To find the Area of the Space $CPBC$; *Fig.* the Curve CPB being the *Logarithmic* 68. *Spiral* *.

Put the ordinate $CB=y$; the length of the curve $CPB=z$; a circular arch, whose radius is the ordinate CB , $=x$; and let two given quantities, a and b , be to each other in the ratio of y to z . Then, (as was found in *art.* 40.) $\dot{x}=y \times \frac{b^2-a^2}{a}^{\frac{1}{2}}$; which, multiplied by $\frac{1}{2}y$, makes $\frac{1}{2}y\dot{x}$ (the general expression

* See the generation of this Curve, *art.* 40. *note.*

174 *An* INTRODUCTION to the
 fion for the Fluxion of the Area, *art.* 124.)

$$= \frac{\sqrt{b^2 - a^2}^{\frac{1}{2}}}{2a} y \dot{y}; \text{ the Eluent of which is}$$

$$\frac{\sqrt{b^2 - a^2}^{\frac{1}{2}}}{4a} y^2 = \text{the Area of the Space CPBC}$$

required.

EXAMPLE V.

Fig. 130. To find the Area of the Space CHBRC;
69. the Curve being a *Spiral* generated by a
70. point moving uniformly along the semi-
 circle CDA, from C to A, while the said
 semicircle makes one uniform revolution
 round the point C as a center *.

Put the radius EC = a , arch CRB = v , or-
 dinate CB = y , and arch DB = x . Then (*art.*

41.) $\dot{x} = \frac{4y\dot{y}}{\sqrt{4a^2 - y^2}^{\frac{1}{2}}}$; which, multiplied by

$\frac{1}{2}y$, makes $\frac{1}{2}y\dot{x}$ (viz. the Fluxion of the Area,
art. 124; for any space generated by the
 arch CRB is equal to that described by the

ordinate CB;) $= \frac{2y^2\dot{y}}{\sqrt{4a^2 - y^2}^{\frac{1}{2}}} = \frac{2y^3\dot{y}}{\sqrt{4a^2y^2 - y^4}^{\frac{1}{2}}}$

$$= \frac{-4a^2y\dot{y} + 2y^3\dot{y}}{\sqrt{4a^2y^2 - y^4}^{\frac{1}{2}}} + \frac{4a^2\dot{y}}{\sqrt{4a^2 - y^2}^{\frac{1}{2}}} = \text{(because,}$$

art.

* This Curve was invented *Anno* 1756.

art. 41. $\dot{v} = \frac{2a\dot{y}}{4a^2 - y^2} \cdot \sqrt{4a^2 - y^2}^{-\frac{1}{2}} \times$
 $\sqrt{4a^2 y\dot{y} - 2y^3\dot{y}} + 2a\dot{v}$; the Fluent of which
 (art. 114. 2^o.) is $-\sqrt{4a^2 y^2 - y^4}^{\frac{1}{2}} + 2av = 2av$
 $- y \times \sqrt{4a^2 - y^2}^{\frac{1}{2}} = GC \times CRB - CB \times BG =$
 four times the Area of the segment CRBC =
 the Area required. And therefore, when the
 point B arrives at G, that is, when $y = 2a$,
 the Area of the whole spiral Space CHBLADC
 will be equal to four times the Area of the
 generating Semicircle.

C H A P. V.

Of finding the Convex Superficies of Solids.

LET the solid AIV be conceived to be ge- Fig.
 nerated by an enlarging or variable circle 71.
 (whose increasing radius is the variable ordi-
 nate of the curve AI,) moving with a pa-
 rallel motion along the axis from A to E:
 then will the velocity with which it's convex
 superficies flows be equal to the periphery of
 the generating circle drawn into the velocity
 with which it moves along the curve AI;
 that is, the Fluxion of the said superficies,
 at

at any term HB, will be equal to the periphery of a circle, whose radius is the ordinate CB, drawn into the Fluxion of the curve at the point B.—This follows from *art.* 124. by considering the convex superficies as always equal to the area of a curvilinear figure whose absciss is equal to the curve AH, and the ordinate as equal to the circumference of the generating circle BH whose radius is CB.

131. HENCE, if we put the absciss $AC=x$, ordinate $CB=y$, curve $AB=z$, and $c=6.28318$ &c. = the circumference of a circle whose radius is 1 (*art.* 122.); then, because cy = the circumference of a circle whose radius is the ordinate CB, and (*art.*

118.) $\dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}^{\frac{1}{2}}$; the General Expression for the Fluxion of the Convex Superficies of any Solid ABH will be $=cy\dot{z}$ or $cyx\sqrt{\dot{x}^2+\dot{y}^2}^{\frac{1}{2}}$; out of which, by help of the Equation of the given Curve AB, $\dot{x}^2$ or $\dot{y}^2$, &c. may be exterminated; and then, the Fluent of the resulting expression will give the Convex Superficies required: As in the following Examples.

EXAMPLE I.

132. To find the Superficies of a *Sphere*, or the Convex Superficies of any Segment of it.

Put the radius EA or $EB=a$, $AC=x$, *Fig.* $CB=y$, and $AB=z$. Let $Bn=Cc$ express 72. the nascent or very first Moment of AC , nb of CB , and Bb of AB ; that is, let $Bn=x'$, $nb=y'$, and $Bb=z'$; then, Bb being considered as a little right line coinciding with a tangent to the point B , the triangles ECB and bnB will be alike: (for, $\angle CBn=\angle Ebb$ $=$ a right angle; therefore, $\angle EBn$ being common, $\angle CBE=\angle nBb$; and, the angles at C and n being right, the angles CEB and Bbn must be likewise equal; *ergo*, &c.) Wherefore, by 4 E. 6. $EB:BC::bB:Bn$, that is, $a:y::z':x'$; $\therefore z'=\frac{ax'}{y}$, or, (*art. 7.*)

$\dot{z}=\frac{a\dot{x}}{y}$; which, substituted for $\dot{z}$, makes the general expression for the Fluxion of the Convex Superficies viz. $cy\dot{z}$ (*art. 131.*) $= cyx\frac{a\dot{x}}{y}=ca\dot{x}$; the Fluent of which is $cax=$

the Convex Superficies of the Segment ABH : And therefore, if for x be substituted $2a$, we

A a

shall

178 *An* INTRODUCTION to the
shall have $2ca^2 =$ the Superficies of the whole
Sphere.

Corollaries.

1. The Convex Superficies of any Segment of a Sphere is equal to the periphery of a great circle of that Sphere multiplied into the altitude of the Segment.

2. The whole Surface or Superficies of any Sphere is equal to the periphery of it's greatest circle multiplied into it's diameter, or, equal to the Convex Superficies of it's circumscribing Cylinder.

EXAMPLE II.

Fig. 133. To find the Convex Superficies of the
73. *Right Cone* AIV, whose Altitude AE is
given $= a$ and base-diameter IV $= b$.

Put $AC = x$, and $CB = y$. Now, the triangles AEI and ACB being alike, by 4 E. 6.

$AE : EI :: AC : CB$; that is, $a : \frac{1}{2}b :: x : y$;

$\therefore x = \frac{2ay}{b}$; the Fluxion of which equation is

$\dot{x} = \frac{2a\dot{y}}{b}$; $\therefore \dot{x}^2 = \frac{4a^2\dot{y}^2}{b^2}$; which, substituted

for $\dot{x}^2$, makes the general expression for the Fluxion of the Convex Superficies viz.

cy

$cy \times \sqrt{x^2 + y^2}^{\frac{1}{2}} \text{ (art. 131.)} = cy \times \sqrt{\frac{4a^2y^2}{b^2} + y^2}^{\frac{1}{2}}$
 $= \frac{c}{b} \times \sqrt{4a^2 + b^2}^{\frac{1}{2}} yy$; the Fluent of which is
 $\frac{c}{2b} \times \sqrt{4a^2 + b^2}^{\frac{1}{2}} y^2 = \frac{cy^2}{b} \times \sqrt{a^2 + \frac{1}{4}b^2}^{\frac{1}{2}} =$ (because
 by 47 E. 1. $AI = \sqrt{AE^2 + EI^2}^{\frac{1}{2}} = \sqrt{a^2 + \frac{1}{4}b^2}^{\frac{1}{2}})$
 $\frac{cy^2}{b} \times AI =$ the Convex Superficies of the Seg-
 ment ABH: And by writing $\frac{1}{2}b$ for y we
 have $\frac{1}{4}cb \times IA =$ the Convex Superficies of the
 Cone AIV.

Corollary.

The Convex Superficies of any Right Cone is equal to half the circumference of it's base multiplied into it's slant height.

EXAMPLE III.

134. To find the Convex Superficies of the *Fig.*
Parabolic Conoid ABH. 74.

Put the parameter of the parabola $= a$,
 $AC = x$, and $CB = y$; then, by the nature
 of the curve, $ax = y^2$; the Fluxion of which
 equation is $a\dot{x} = 2y\dot{y}$; therefore, $\dot{x} = \frac{2y\dot{y}}{a}$, and

Aa 2

$\dot{x}^2 =$

$\dot{x}^2 = \frac{4y^2 \dot{y}^2}{a^2}$; which, substituted for $\dot{x}^2$, makes the general expression for the Fluxion of the Convex Superficies viz. $c y \times \overline{x^2 + y^2}^{\frac{1}{2}}$ (*art.*

$$131.) = c y \times \overline{\frac{4y^2 \dot{y}^2}{a^2} + y^2}^{\frac{1}{2}} = \frac{c}{a} \times \overline{4y^2 + a^2}^{\frac{1}{2}} y \dot{y};$$

the Fluent of which (*art.* 114.) is $\frac{c}{12a} \times$

$\overline{4y^2 + a^2}^{\frac{3}{2}}$: But, at the vertex A, where y vanishes or $y=0$, this fluential expression becomes $= \frac{c}{12a} \times \overline{a^2}^{\frac{3}{2}} = \frac{ca^2}{12}$; therefore, the

Fluent corrected (*art.* 117.) is $= \frac{c}{12a} \times$

$\overline{4y^2 + a^2}^{\frac{3}{2}} - \frac{ca^2}{12} =$ the Convex Superficies of

the Solid ABH required.

C H A P. VI.

Of finding the Contents of Solids.

Fig. 135. LET bc be conceived indefinitely
75. near and parallel to the variable ordinate
BC; and Bn equal and parallel to Cc the
Increment of the variable absciss AC. Now,
be-

because the little parallelogram $BCcn$ is expressive of the Increment or Moment of the plane figure ABC , (*art.* 124.) therefore, if the solid AIV be conceived to be generated by the revolution of the curvilinear figure AEI round the axis AE , the indefinitely little cylinder generated by the said little parallelogram will express the Moment or Increment of the solid at the term BH ; and, because this Moment or Increment is equal to the area of the circle described by the ordinate CB drawn into the Increment of the absciss AC , therefore, (*art.* 7.) the Fluxion of the Solid, at the term BH , is equal to the area of a circle, whose radius is the ordinate CB , drawn into the Fluxion of the absciss AC .

Or, Let the cylinder LG be generated by the parallel motion of the circle LD along the line LN ; and, at the same time, the solid AIV by that of the concentric and variable circle ZF , which, at the vertex A , is supposed indefinitely small, and continually enlarges as it moves along the curve AI . Then, it is plain, the solid AIV will increase slower, or flow with a less degree of velocity, than the cylinder LG , before the generating circles arrive at the term HB ; and after-

afterwards faster, or with a greater degree of velocity: therefore, at the said term, (where the two generating circles become equal, or their peripheries coincide with each other,) they will increase, or flow, with the same or an equal degree of velocity. But, it is evident, the velocity with which the cylinder flows is equal to the area of it's generating circle drawn into the velocity with which it moves along the line LN; that is, the Fluxion of the cylinder, at the term HB, is equal to the area of a circle, whose radius is CB, drawn into the Fluxion of the line LH or AC: Therefore, the Fluxion of the Solid AIV, at the term BH, is equal to the area of a circle, whose radius is the ordinate CB, drawn into the Fluxion of the absciss AC; as before.

136. HENCE, if we put the absciss $AC=x$, ordinate $CB=y$, and $c=3.14159 \&c.$ = the semicircumference of a circle whose radius is 1 (*art.* 122.); then, the General Expression for the Fluxion of the Solid Content will be $=cy^2\dot{x}$: out of which, by help of the Equation of the given Curve, $\dot{x}$ or y^2 may be exterminated; and then, by finding the Fluent of the resulting fluxional expression, we shall have the Content of the Solid ABH required.

E x-

EXAMPLE I.

137. To find the Content of a *Sphere*, or of any *Segment* of it.

Put $AC=x$, $CB=y$, and the diameter *Fig.* $AD=a$; then, by 35 E. 3. $AC \times CD = BC \times CH$, that is, $ax - x^2 = y^2$; therefore, by writing $ax - x^2$ for y^2 , the general expression for the Fluxion of the Solid Content viz. $cy^2 \dot{x}$ (*art.* 136.) becomes $= c\dot{x} \times ax - x^2 = cax\dot{x} - cx^2\dot{x}$; the Fluent of which is $\frac{cax^2}{2} - \frac{cx^3}{3} = \frac{3cax^2 - 2cx^3}{6}$ = the Content of the Segment ABH: And therefore, if a be substituted for x , we shall have $\frac{3ca^3 - 2ca^3}{6} = \frac{1}{6}ca^3$ = the Content of the whole Sphere ABDH.

Hence, because four times the area of a great circle of the sphere is $= ca^2$, and the content of a cylinder circumscribing the sphere is $= \frac{1}{4}ca^3$, we have the following

Corollary.

The Content of any Sphere is equal to four times the area of it's greatest circle multiplied into $\frac{1}{6}$ th part of it's axis, or,
equal

equal to two-third-parts of it's circumscribing cylinder.

EXAMPLE II.

Fig. 138. To find the Content of the *Parabolic*
74. *Conoid* ABH, generated by the parabolic
 space ABC revolving round the axis AC.

Put the parameter $=a$, $AC=x$, and $CB=y$; then, by the nature of the curve, $ax=y^2$. Now, by substituting ax for y^2 , we have the general expression for the Fluxion of the Solid Content viz. $cy^2\dot{x}$ (*art.* 136.) $=cax\dot{x}$; the Fluent of which is $\frac{1}{2}cax^2$ (by writing y^2 for ax), $\frac{1}{2}cxy^2$ = the Content of the Parabolic Conoid ABH required.

Or, The Fluxion of the equation of the curve, viz. of $ax=y^2$, is $a\dot{x}=2y\dot{y}$; therefore $\dot{x}=\frac{2y\dot{y}}{a}$; which, substituted for $\dot{x}$, makes the general expression for the Fluxion of the Solid Content viz. $cy^2\dot{x}$ (*art.* 136.) $=cy^2 \times \frac{2y\dot{y}}{a} = \frac{2cy^3\dot{y}}{a}$; the Fluent of which is $\frac{cy^4}{2a}$ = (by writing ax for y^2), $\frac{1}{2}cxy^2$ = the Solid Content; as before.

Corol.

Corollary.

The Content of any Parabolic Conoid is equal to half of it's circumscribing Cylinder.

EXAMPLE III.

139. To find the Content of *any Cone* AIV *Fig.*
whose base is a circle. 77.

Put the given altitude $AE=a$, and diameter $VI=b$. Let HB be parallel to VI ; and put $AC=x$, and $c=.78539$ &c. = the area of a circle whose diameter is 1. Now, the triangles AVI and AHB being similar, by 4 E. 6. $AE : VI :: AC : HB$, that is, $a :$

$b :: x : \frac{bx}{a} = HB$; therefore, by 2 E. 12. the

area of the circle HB is $= \left[\frac{bx}{a} \right]^2 \times c = \frac{cb^2x^2}{a^2}$;

which (*art.* 135.) drawn into $\dot{x}$ is $\frac{cb^2x^2\dot{x}}{a^2} =$

the Fluxion of the Content of the Cone at the term CB ; the Fluent of which is $\frac{cb^2x^3}{3a^2}$

= the Content of the Cone AHB : And, therefore, by substituting a for x , we have

$\frac{cb^2a^3}{3a^2} = \frac{1}{3}cb^2a =$ the Content of the Cone

AVI required.

B b

Or,

Or, Put the altitude $AE=a$, the area of the base $VI=b$, and $AC=x$: then, (because similar plane Figures are as the squares of their homologous sides,) we shall have

$$a^2 : b :: x^2 : \frac{bx^2}{a^2} = \text{the area of the section}$$

HB; which (*art.* 135.) drawn into $\dot{x}$ is $\frac{bx^2\dot{x}}{a^2}$

= the Fluxion of the Cone AHB; the Fluxion of which expression gives the Content of

the said Cone $= \frac{1}{3} \times \frac{bx^3}{a^2}$: And therefore, by

writing a for x , we have the Content of the whole Cone $AVI = \frac{1}{3}ab$. Hence, because b may here stand for the area of the base of any Pyramid whatever, we have the following

Corollary.

The Solid Content of any Cone, or Pyramid, is equal to the area of it's base multiplied into one-third-part of it's perpendicular altitude; that is, it is equal to one-third-part of a Cylinder, or Prism, of the same altitude and base.

E x-

EXAMPLE IV.

140. To find the Content of the Solid ABH, *Fig.*
generated by the *Cissoïdal* Space ABC re- 78.
volving round the axis AE *.

Put the axis $AE=a$, absciss $AC=x$, and
ordinate $CB=y$; then, (as was found in
art. 34.) $x^3=ay^2-xy^2$; and therefore, $y^2=$

$\frac{x^3}{a-x}$; which, substituted for y^2 , makes the

general expression for the Fluxion of the
Solid Content viz. $cy^2\dot{x}$ (*art.* 136.) $= \frac{cx^3\dot{x}}{a-x}$

$= \frac{-cx^3\dot{x}}{x-a} = -cx^2\dot{x} - cax\dot{x} - ca^2\dot{x} - \frac{ca^3\dot{x}}{x-a}$

$= -cx^2\dot{x} - cax\dot{x} - ca^2\dot{x} - ca^3 \times \frac{-\dot{x}}{a-x}$; the Flu-

ent of which, (because *art.* 21. the Fluxion
of the Hyp. Log. of $a-x$ is $\frac{-\dot{x}}{a-x}$,) is $= -$

$\frac{cx^3}{3} - \frac{cax^2}{2} - ca^2x - ca^3 \times \text{Hyp. Log. of } a-x.$

But, when $x=0$, (as at A,) this Fluent be-
comes $= -ca^3 \times \text{Hyp. Log. of } a$: there-
fore, the Fluent corrected (*art.* 117.) is $=$

* See how a *Cissoïd* is generated, *art.* 34. *note.*

$$\frac{cx^3}{3} - \frac{cax^2}{2} - ca^2x - ca^3 \times \text{Hyp. Log. of}$$

$$\frac{a-x}{a-x} + ca^3 \times \text{Hyp. Log. of } a = \text{(because the}$$

Difference of the Logarithms of any two numbers is equal to the Logarithm of their Quotient,) $-cx \times \frac{2x^2 + 3ax + 6a^2}{6} + ca^3 \times$

Hyp. Log. of $\frac{a}{a-x} =$ the Content of the Solid ABH required.

Corollary.

When $x = \frac{1}{2}a$, the Content of the Solid will be $= ca^3 \times : -\frac{2}{3} + \text{Hyp. Log. of } 2 = ca^3 \times 0.02648 \text{ \&c.}$ (See *Part III. Quest. 9.*)

P A R T I I I.

MISCELLANEOUS QUESTIONS, With their *Incremental* and *Fluxional* SOLUTIONS.

I.

In an *Ellipsis* ABD, whose Foci are the points F and K, if a right line Bt be drawn bisecting the angle FBK; then will the said line be perpendicular to the tangent TBG. *Quære* the Demonstration? Fig:
79.

Let the point b be supposed indefinitely near to B; and with the lines FB and Kb, as radii, describe the arches Bm and bn: then, if we consider the said arches as little right lines perpendicular to Fb and KB respectively, (because the Sum of the lines FB and KB is always the same invariable quantity, viz. = AD, and therefore the Increment mb

=

=the Decrement nB ,) the right angled triangles BnO and bmO will be equal and similar, as will therefore the right angled triangles Emr and bnr ; and therefore, if Bb , the Increment of the curve AB , be supposed to coincide with the tangent BG , the angles rbB and rBb will be equal, that is, $\angle FBT = \angle KBG$. Therefore, the right line Bt , which bisects the $\angle FBK$, makes the $\angle tBT = \angle tBG =$ a Right angle. Q. E. D.

II.

Fig. In an *Hyperbola* AB , whose Focus is the point F , and transverse Diameter is $DA = KA - AF$; the right line tB , which bisects the angle KBF , is a tangent to the curve at the point B . *Quære* the Demonstration?

Suppose the point b indefinitely near to B , and describe the little arches Bm and Bn with the radii KB and FB . Now, because the Difference of the lines KB and FB is always the same invariable quantity, viz. $= DA$, the Increments mb and nb are equal; and therefore, if the arches Bm and Bn be considered as little right lines perpendicular to Kb and Fb respectively, and Bb (the Increment of the curve AB) as coinciding with the

the

the tangent; the little right angled triangles Bmb and Bnb will be equal and similar. Therefore, the right line tB , bisecting the angle KBF , is a tangent to the curve at the point B . Q. E. D.

III.

In a *Parabola* AB , whose Focus is the point *Fig.*
 F , if a right line KB be drawn parallel to 81.
the axis, and the angle KBF be bisected
by the right line tB ; then will this line
be a tangent to the curve at the point B .
Quære the Demonstration?

Let the indefinite right line LK be perpendicular to LF , and $LA=AF$; draw kb indefinitely near and parallel to KB , and Bm equal and parallel to Kk ; and with FB , as a radius, describe the little arch Bn . Now, because the lines KB and FB are always equal to each other, the Increment mb is = the Increment nb ; and therefore, (the indefinitely small arch Bn being considered as a little right line perpendicular to Fb , and the Increment of the curve, Bb , as coinciding with the tangent,) the little right angled triangles Bmb and Bnb are equal and similar. Consequently, the right line tB , which bisects

sects the angle KBF, is a tangent to the curve at the point B. Q.E.D.

IV.

Quære the Nature of the Curve APB?—

Fig. supposing FP or CE, the distance of the
82. parallel and indefinite right lines AC and PE, to be given; the right line AD to pass through the point P; and $CE \times CD = CB^2$. *

Put $CE = a$, absciss $AC = x$, ordinate $CB = y$, and $CD = z$. Let cd be supposed indefinitely near and parallel to CD ; and Dm , Bn , equal and parallel to Cc : and put $nb = y'$, $md = z'$; and suppose TB a tangent to the curve at the point B. Now, by 4 E. 6.

$DC : CA :: dm : mD$, that is, $z : x :: z' : \frac{xz'}{z}$
 $= Dm$ or Bn ; and $bn : nB :: BC : CT$, that
is, $y' : \frac{xz'}{z} :: y : \frac{xyz'}{zy'} = CT$, or (art. 7.) $\frac{xyz'}{zy'}$

$= CT$. But, by the question given, $az = y^2$;
the Fluxion of which equation is $a\dot{z} = 2y\dot{y}$,

∴

* To describe the Curve, or to find the Point B in the line CD through which it must pass.—Produce CD to G, making $DG = FP$; describe the semicircle GHC, and draw the perpendicular ordinate DH; lastly, make $CB = DH$: then will B be the Point required. For, by 35 E. 3. $GD \times DC = DH^2$; that is, $EC \times CD = CB^2$.

$\therefore \dot{z} = \frac{2y\dot{y}}{a}$; which, substituted for $\dot{z}$, makes the above value of the Subtangent CT (viz. $\frac{xy\dot{z}}{\dot{y}} = \frac{2xy^2}{az} = (\text{because } \frac{y^2}{az} = 1,) \quad 2x$. Therefore, (*art.* 28.) the curve AB is a *Parabola*, whose vertex is A.

Corollary.

If F be the Focus; then, by the nature of the Parabola, $PF = 2FA$; and therefore $DC = 2CA = CT$, that is, $z = 2x$.

V.

If TB be a Tangent to the given Curve AB; *Fig.* and another Curve AD be so described as 83. that it's ordinate CD shall always be in a given ratio to the corresponding Segment of the former Curve: then, BT will be to TC as the corresponding Segment of the Curve AB is to the Subtangent CV. *Quære* the Demonstration?

Put $CT = s$, $TB = t$, $CD = y$, $AB = z$: and let cd be supposed indefinitely near and parallel to CD; and Dm , Bn , equal and parallel to Cc ; that is, let $md = y'$, and $Bb = z'$. Now, $BT : TC :: bB : Bn$, that is, $t : s ::$

C c
 z'

$$z' : \frac{sz'}{t} = Bn \text{ or } Dm; \text{ and } dm : mD :: DC : CV,$$

$$\text{that is, } y' : \frac{sz'}{t} :: y : \frac{sy'z'}{ty'} = CV, \text{ or (art. 7.)}$$

$$\frac{sy\dot{z}}{ty} = CV. \text{ Let the given ratio of } DC \text{ to}$$

$$AB \text{ be as } a \text{ to } b, \text{ that is, } y : z :: a : b, \therefore z = \frac{by}{a}; \text{ the Fluxion of which equation is } \dot{z} =$$

$$\frac{b\dot{y}}{a}; \text{ which substituted for } \dot{z} \text{ makes the above}$$

$$\frac{sy\dot{z}}{ty} = CV = \frac{bsy}{at} \text{ (which, by writing } z \text{ for it's}$$

$$\text{value } \frac{by}{a}, \text{ is) } = \frac{sz}{t}; \text{ therefore, } t : s :: z : CV,$$

$$\text{that is, } BT : TC :: AB : CV. \text{ Q. E. D.}$$

VI.

Fig. In the Curve ABD, whose Equation (putting the absciss $AC=x$, ordinate $CB=y$, and the base $AD=a$,) is $ax-x^2=ay+y^2$: required the Radius of Curvature for any point, and a geometrical Construction to illustrate and confirm the Work.

The Fluxion of the given equation of the curve is $a\dot{x}-2x\dot{x}=\dot{a}y+2y\dot{y}$; which, making $\dot{x}=1$, is $a-2x=\dot{a}y+2y\dot{y}$; and

and the Fluxion of this equation again, the Fluxion of $\dot{y}$ being considered as negative, is $-2 = -a\ddot{y} + 2\dot{y}^2 - 2y\ddot{y}$. Hence we have

$$\dot{y} = \frac{a-2x}{a+2y}, \dot{y}^2 = \frac{a^2-4ax+4x^2}{a^2+4ay+4y^2}, \text{ and } \ddot{y} = \frac{4a^2+8ay+8y^2-8ax+8x^2}{(a+2y)^3} =$$

(because by the equation of the curve, $8ay+8y^2=8ax-8x^2$, or $8ay+8y^2-8ax+8x^2=0$.)

$$\frac{4a^2}{(a+2y)^3}. \text{ Now, by substituting for } \dot{y}^2 \text{ and } \ddot{y}$$

these their values, in the general expression for the Radius of Curvature, which was found

in *art.* 74. to be $= \frac{\sqrt{1+\dot{y}^2}^{\frac{3}{2}}}{\dot{y}}$ when $\dot{x}=1$ and

the Fluxion of y is negative, we shall have

$$\frac{(2a^2+4ay+4y^2-4ax+4x^2)^{\frac{3}{2}}}{(a+2y)^2} \times \frac{(a+2y)^3}{4a^2} =$$

$$(\text{because } 4ay+4y^2-4ax+4x^2=0,) \frac{(2a^2)^{\frac{3}{2}}}{(a+2y)^2}$$

$$\times \frac{(a+2y)^3}{4a^2} = \frac{(2a^2)^{\frac{3}{2}}}{4a^2} = \frac{8^{\frac{1}{2}}a}{4} = \frac{1}{2}a = \text{the Ra-}$$

dius of Curvature required; which being a fixed or invariable quantity proves the curve to be an *Arch* of a *Circle*.

Now, if the Radius of a Circle be $\sqrt{\frac{1}{2}}a$; then, a is the side of it's inscribed square, or the chord of 90° . as is the right line AD: For, if the said right line AD be bisected in F, and the perpendicular FE be drawn $= AF = \frac{1}{2}a$; the point E will be the center of the circle; and consequently, the radius will be AE ($= \sqrt{EF^2 + FA^2}$) $= \sqrt{\frac{1}{2}a^2} = \sqrt{\frac{1}{2}}a$. And, that x , in the given equation, must flow in the said chord AD, may be thus demonstrated: Draw the right line BC perpendicular to AD, and let it be produced until it meets the circle's periphery in G; and let HI be drawn equal and parallel to AD: then, it is evident, that, CK=AH=AD, by construction; and KG=CB, because AC=HK, and AB=HG; therefore, CG=AD+CB: But, by 35 E. 3. AC×CD=BC×CG, that is, (putting AD= a , AC= x , and CB= y ,) $x \times a - x = y \times a + y$, or $ax - x^2 = ay + y^2$. Therefore, &c.

SCHOLIUM.

From the Construction here given, it appears, that, the Question may be diversified so as to be adapted to any regular Polygon that can be inscribed in a Circle. We see here

here also, a demonstration of the justness of the *fluxional* Calculus as made use of above.

VII.

Suppose the Earth to revolve in a circular orbit round the Sun as it's center; and the Moon to revolve round the Earth in the same manner; as also, that the planes of their orbits do coincide; and that the diameters of the said orbits are as 340 to 1; and lastly, that the Moon performs 13.368 revolutions to every single revolution of the Earth. *Quære* the Nature and Description of the Curve generated by the Center of the Moon; or, whether the Curve described by the Center of the Moon, in one Luration, be any-where *Convex* towards the Sun?

Let S represent the Sun; E, the Earth; *Fig.*
E, an arch of the orbit of the Earth passed 85.
 over by it's center in one luration of the
 Moon; the circumference of the circle EAF
 = the concentric arch Aa: Then, (because
 $13.368 - 1 = 12.368$ = the number of luna-
 tions in a year or one revolution of the
 Earth, and therefore $SA : EA :: 12.368 : 1$.)
 when the Moon is in conjunction with the
 Sun, the distance between the Sun and Moon
 will

will be greater than the distance or radius SA. Now, the Curve described by the center of the Moon is the same as that described by a point M (EM being the semidiameter of the Moon's orbit,) carried round by the rotation of the circle EAF on the arch Aa: It is therefore of the *Cycloidal* Kind, having a point of Inflection, if every Cycloid described by a point within the generating circle is inflected as well upon a *circular* as upon a *rectilinear* base (*art.* 65.). To determine which,

Put SA or SR = a , EA or eR = b , EM or em = c , Rm = r , Rd = s ; and let mC be the Radius of Curvature at any point m , which, it is evident, must pass through the point of contact R. Suppose the point n indefinitely near to m : Then, Rr and Rr being the indefinitely small contemporary arches with mn , and consequently the triangles Rmr and Rnr equal in all respects; if we consider the said little arches Rr and Rr as little right lines perpendicular to the radii er and Sr, we shall have the $\angle mRn = \angle rRr =$ (because the angles eRr and SRr added to either side of the equation make it two right angles,) $\angle Rer + \angle RSr$. Now, SR : eR :: $\angle Rer$: $\angle RSr$, and SR : SR + Re :: $\angle Rer$: $\angle Rer + \angle RSr$,

$\angle R S r$, that is, $a : a+b :: \angle R e r : \angle m R n = \frac{a+b}{a} \angle R e r$. Again, in any triangle, as $d m r$, if the angles $m d r$, $m r d$, and $R m r$ the complement of the obtuse angle to two right angles, be indefinitely small, they will be proportional to the opposite sides $m r$, $m d$, and $d r$ *, that is, $d r : m d :: \angle R m r : \angle m r d$; and $d r - m d : d r :: \angle R m r - \angle m r d : \angle R m r$, that is, $m R : d R :: \angle R d r : \angle R n r$, or, $r : s :: \frac{1}{2} \angle R e r : \angle R n r = \frac{s}{2r} \angle R e r$. And again, $\angle R C n : \angle R n C :: R n : R C$, that is, $\angle m R n - \angle R n r : \angle R n r :: R m : R C$, or, $\frac{a+b}{a} - \frac{s}{2r} \angle R e r : \frac{s}{2r} \angle R e r :: r : R C = \frac{ars}{2ar + 2br - as}$. Consequently, $m R + R C = m C = \frac{2ar^2 + 2br^2}{2ar + 2br - as} = r^2$

* For, let the triangle be circumscribed (as in *fig. 86.*) by the circle $r m d$: then will the arches $d m$ and $m r$ differ infinitely little from their chords $d m$ and $m r$, which therefore may be taken as equal to them. And, since by 20 E. 3. the angle at the center of a circle is double of the angle at the periphery, the arches or chords $r m$ and $m d$ are the measures of double the angles $m d r$ and $m r d$ respectively; or, the angles $m d r$ and $m r d$ are to each other as their opposite sides $r m$ and $m d$: and because by 32 E. 1. the angle $R m r$ is equal to the sum of the angles $m d r$ and $m r d$; therefore the measure of it is the sum of the arches or chords $r m$ and $m d$; which differing infinitely little from the side or chord $r d$, may be considered as equal to it. *Ergo*, &c.

$\frac{r^2}{r - \frac{as}{2a+2b}}$ = the Radius of Curvature at any point m .

Now, it is evident, that, at the point of Inflection, the Radius of Curvature must be Infinite; or, that, on one side of the said point, the expression for the radius of curvature must be affirmative, and on the other negative; therefore, r must be more than

$\frac{as}{2a+2b}$ on one side of the said point, and on the other less; and consequently, at the point

of Inflection, $r = \frac{as}{2a+2b}$; which, substituted for r , makes ($dm \times mR =$) $rs - r^2 =$

$\frac{2abs^2 + a^2s^2}{(2a+2b)^2} =$ (because $dm \times mR = fm \times ma$
 $=$) $b^2 - c^2$; from which equation we have

$s = \frac{2a+2b\sqrt{b^2-c^2}}{\sqrt{2ab+a^2}}$. — Or, to find r , say

$2ar + 2br = as$, or $s = \frac{2ar+2br}{a}$; then, (dm

$\times mR =$) $rs - r^2 = \frac{ar^2+2br^2}{a} = (fm \times$

$ma =$) $b^2 - c^2$; which equation gives r

=

$= \sqrt{\frac{ab^2 - ac^2}{a + 2b}}$, when the point m becomes a point of Inflection.

Now, as mR (r) must, by the nature of the circle, always be greater than ma ; that is, as $\sqrt{\frac{ab^2 - ac^2}{a + 2b}}$ must always be more

than $b - c$; and consequently, $\frac{ab^2 - ac^2}{a + 2b}$ be

more than $\overline{b - c}^2$, that is, $\frac{ab + ac}{a + 2b} \times \overline{b - c}$ be

more than $\overline{b - c} \times \overline{b - c}$; therefore, c must always be more than $\frac{b^2}{a + b}$; that is, EM must

be *more* than a third proportional to ES and EA in order to have a point of Inflection take place in the Curve: But, in the present case, ES , EA , and EM , being as 13.368, 1, and $\frac{13.368}{340}$ or .039; therefore, EM is *less*

than the said third proportional; and consequently, the Curve $Mm\mu$, generated by the Center of the Moon, has *not* a point of Inflection, or, is *no-where* Convex towards the Sun. Q.E.I.

Corollaries.

1. When $r=s$, that is, when the point m coincides with a , or a is the generating point; then, the Radius of Curvature will be $= \frac{2ar^2 + 2br^2}{ar + 2br} = \frac{a+b}{a+2b} 2r$; and $RC = \frac{a}{a+2b} r$; by analogy, $a+2b : a :: r : RC$, that is, $SF : SA :: mR : RC$.

2. When a is Infinite, and $r=s$; that is, when the base becomes a right line, and the point m coincides with a , or the curve is the common Cycloid; the Radius of Curvature will be $= 2r$. For then, $2br^2$ and $2br$ will be infinitely little in comparison of $2ar^2$ and ar , and therefore may be rejected.

Fig. 3. When a is Infinite, that is, when the
87. base degenerates into a right line, or the curve is the protracted or interior Cycloid, as in *fig.* 87. the Radius of Curvature will become $= \frac{2ar^2}{2ar - as} = \frac{2r^2}{2r - s}$; and therefore, at the point of Inflection, where the radius of curvature is infinite, $2r=s$, that is, $Rm=md$; and consequently the rightline Rd is perpendicular to the radius ea , and the point a is in the base NM . Whence, to find the point of Inflection, we have the following
Con-

Construction, viz. Make $PR=ag$, or $AR=Pg$; draw Re equal and parallel to the radius PO ; make $ra=Ng$; draw the right line ea ; and, lastly, make $em=ON$: Then will m be the point of Inflection in the interior Semicycloid $_{m}M$.

VIII.

The *Fluxion* of the Hyperbolic Logarithm of any quantity, is equal to the Fluxion of that quantity, divided by the quantity itself. *Quære* the Demonstration? (See *art. 21.*)

Let YAI be an Hyperbola, whose asymp- *Fig.*
totes are the perpendicular right lines EZ 88.
and ET , and whose parameter is $AP=EP$
 $=1$; draw any ordinate CB parallel to PA :
then, (as Writers on *Conics* demonstrate,) the
Space $PABC$ will be the Hyperbolic Lo-
garithm of the line EC ; and therefore, the
Fluxion of the space $PABC$ will be equal to
the Fluxion of the Hyperbolic Logarithm
of the line EC . Now, the Fluxion of this
space, putting $EC=x$ and $CB=y$, is (by
art. 124.) $=y\dot{x}$; and, by the known proper-
ty of the curve, $EC : EP :: PA : CB$, that
is, $x : 1 :: 1 : y = \frac{1}{y}$: therefore $y\dot{x} = \frac{\dot{x}}{y}$; that
D d 2 is,

is, the Fluxion of the space PABC, or, of the Hyperbolic Logarithm of the line EC, is equal to the Fluxion of the said line, divided by the line itself. Q. E. D.

N. B. By a Space or Line, is meant it's Numerical Measure.

IX.

The Hyperbolic Logarithm of 1 being 0, what is the Hyperbolic Logarithm of 10?
(See *art.* 21.)

Fig. 89. Let EZ and ET be the asymptotes of the rectangular Hyperbola YAI, whose parameter is $AP=PE=1$. Then, the area of the space PABC will be the Hyperbolic Logarithm of $\frac{EC}{EP}$, that is, of EC; the area of the space PA $\hat{b}$ c will be the Hyperbolic Logarithm of $\frac{EP}{Ec}$, that is, of $\frac{1}{Ec}$; and, the area of the space c $\hat{b}$ BC will be the Hyperbolic Logarithm of $\frac{EC}{Ec}$; the right lines or ordinates CB and c $\hat{b}$ being supposed parallel to the asymptote EZ. Now, to find these areas,

1°. Put $Pc=x$, and $cb=y$; then, by the known property of the curve, $y=\frac{1}{1-x}$;
which,

which, (*art.* 124.) drawn into $\dot{x}$, is $y\dot{x} =$

$\frac{\dot{x}}{1-x} =$ the Fluxion of $Pb =$ (by throwing

$\frac{\dot{x}}{1-x}$ into a Series, *art.* 102.) $\dot{x} + x\dot{x} + x^2\dot{x}$

$+ x^3\dot{x} + x^4\dot{x} + x^5\dot{x} + x^6\dot{x} + x^7\dot{x} + x^8\dot{x} + x^9\dot{x}$
 $+ x^{10}\dot{x} + x^{11}\dot{x} + \mathcal{E}c.$ therefore the Fluent of
 this series, viz. $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 +$
 $\frac{1}{6}x^6 + \frac{1}{7}x^7 + \frac{1}{8}x^8 + \frac{1}{9}x^9 + \frac{1}{10}x^{10} + \frac{1}{11}x^{11} +$
 $\frac{1}{12}x^{12} + \mathcal{E}c.$ is $= Pb.$

2°. Put $PC = x$, and $CB = y$; then, $y =$

$\frac{1}{1+x}$, and (*art.* 124.) $y\dot{x} = \frac{\dot{x}}{1+x} =$ the Flu-

xion of $PB =$ (by throwing $\frac{\dot{x}}{1+x}$ into a Se-

ries, *art.* 103.) $\dot{x} - x\dot{x} + x^2\dot{x} - x^3\dot{x} + x^4\dot{x} -$
 $x^5\dot{x} + x^6\dot{x} - x^7\dot{x} + x^8\dot{x} - x^9\dot{x} + x^{10}\dot{x} - x^{11}\dot{x} +$
 $\mathcal{E}c.$ therefore the Fluent of this series, viz.
 $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \frac{1}{7}x^7 - \frac{1}{8}x^8$
 $+ \frac{1}{9}x^9 - \frac{1}{10}x^{10} + \frac{1}{11}x^{11} - \frac{1}{12}x^{12} + \mathcal{E}c.$ is $=$
 $PB.$

Hence,

If $Pc = PC$, the area $cbBC$, viz. the sum
 of Pb and PB , will be $= 2x : x + \frac{1}{3}x^3 + \frac{1}{5}x^5$
 $+ \frac{1}{7}x^7 + \frac{1}{9}x^9 + \frac{1}{11}x^{11} + \mathcal{E}c.$ And $Pb - PB$
 will be $= x^2 + \frac{1}{2}x^4 + \frac{1}{3}x^6 + \frac{1}{4}x^8 + \frac{1}{5}x^{10} + \frac{1}{6}x^{12} +$
 $\mathcal{E}c.$ That is, if $Ec = .9$, and $cP = PC = .1$;
 then,

then, by substituting .1 for x , we shall have
 $cB = .2006706954 \text{ } \&c.$ and $Pb - PB =$
 $.0100503358 \text{ } \&c.$ half of which added to
half cB is $.1053605156 \text{ } \&c. = Pb =$ the Hyp.
Log. of $\frac{1}{.9}$. And, if $Ec = .8$, and $cP = PC =$
 $x = .2$; then, by writing .2 for x , we have cB
 $= .4054651081 \text{ } \&c. =$ the Hyp. Log. of $\frac{1.2}{.8}$;
and $Pb - PB = .0408219945 \text{ } \&c.$ half of
which subtracted from half cB is $.1823215567$
 $\&c. = PB =$ the Hyp. Log. of $\frac{1.2}{1}$ or 1.2;
and this subtracted from cB leaves $.2231435513$
 $\&c. = Pb =$ the Hyp. Log. of $\frac{1}{.8}$. Therefore,
by the Nature of Logarithms, the Hyp. Log.
of $\frac{1}{.9} \times \frac{1.2}{.8} \times 1.2$, viz. of 2, is $= .1053605156$
 $\&c. + .4054651081 \text{ } \&c. + .1823215567$
 $\&c. = .6931471805 \text{ } \&c.$ and, the Hyp. Log.
of 2^3 , viz. of 8, is $= 3 \times .6931471805 \text{ } \&c.$
 $= 2.0794415416 \text{ } \&c.$ and, the Hyp. Log. of
 $8 \times \frac{1}{.8}$, viz. of 10, is $= 2.0794415416 \text{ } \&c.$
 $+ .2231435513 \text{ } \&c. = 2.3025850929 \text{ } \&c.$
Q. E. I.

X.

Let the given right line CA turn uniformly round the point C as a center; and, let a point be supposed to pass with an uniform motion, from A, along the right line AF, equal and perpendicular to the said line CA, and such velocity, as to arrive at F at the same time that the said lines come to be in their first situation: then, by this point, will the *Spiral* APBF be described. *Quære* the Area of any Space ARaBPA? *

Put $CA=AF=a$, the circumference of the circle $ARaA=b$, $aB=v$, arch DB (described with the ordinate or radius CB,) $=x$, $CB=y$, arch $ARa=z$, $Bn=x'$, and $aa=z'$. Now, (supposing Bn , the Increment of the arch DB, to be a little right line perpendicular to the radius CB,) if we draw aG perpendicular to CB, the Moment or Increment of the space $ARaBD$, viz. $aBnaa$, will, by 41 E. 1. be $=\frac{1}{2}GB \times Bn =$ (because by 8 and 4 E. 6. $CB : Ba :: aB : BG$, or, $y : v :: v : BG = \frac{v^2}{y}$,) $\frac{v^2 x'}{2y}$; therefore, art. 7. the

Fluxion

* This Curve was invented Anno 1756.

Fluxion of the said space, which is = the Fluxion of the space in question, is $= \frac{v^2 \dot{x}}{2y}$.

But, it is plain from the generation of the Curve, (af) $a : b :: \dot{v} : \dot{z} = \frac{b\dot{v}}{a}$, and (Ca) $a :$

$y :: \dot{z} : \dot{x} = \frac{y\dot{z}}{a} = \frac{y}{a} \times \frac{b\dot{v}}{a} = \frac{by\dot{v}}{a^2}$; which substituted for $\dot{x}$ makes the above Fluxion of the space in question $= \frac{bv^2\dot{v}}{2a^2}$; the Fluent of

which is $\frac{bv^3}{6a^2} =$ the Area of the Space required: And therefore, by writing a for v , we shall have the Area of the whole spiral Space ARAFBPA $= \frac{1}{6} ab = \frac{1}{3}$ the Area of the Circle CARaA. Q. E. I.

Or, By 47 E. I. $v = \sqrt{y^2 - a^2}^{\frac{1}{2}}$; the Fluxion of which equation is $\dot{v} = \frac{y\dot{y}}{\sqrt{y^2 - a^2}^{\frac{1}{2}}}$; therefore

$\dot{x} (= \frac{by\dot{v}}{a^2}) = \frac{b}{a^2} \times \frac{y^2\dot{y}}{\sqrt{y^2 - a^2}^{\frac{1}{2}}}$; which substituted

for $\dot{x}$, makes $\frac{1}{2} y\dot{x}$ viz. the Fluxion of the Area art. 124. (for CB and CaB describe equal Spaces,)

$$= \frac{b}{2a^2} \times \frac{y^3\dot{y}}{\sqrt{y^2 - a^2}^{\frac{1}{2}}} = \frac{b}{6a^2} \times \frac{3y^5\dot{y} - 2a^2y^3\dot{y}}{\sqrt{y^6 - a^2y^4}^{\frac{1}{2}}} + \frac{1}{3}b$$

$\frac{1}{3}b \times \frac{y\dot{y}}{y^2 - a^2}^{\frac{1}{2}}$; the Fluent of which expres-

sion is $\frac{b}{6a^2} \times y^6 - a^2 y^4^{\frac{1}{2}} + \frac{1}{3}b \times y^2 - a^2^{\frac{1}{2}} =$

the Area of the Space ACaBPA. So that, by writing $2a^2$ for y^2 , we have the Area of the Space AFBPA $= \frac{2}{3}ab$; from which if we take $\frac{1}{2}ab$ (the Area of the Circle ARA,) there will remain $\frac{1}{6}ab =$ the Area of the whole spiral Space; as before.

Corollary.

$$\frac{\dot{x}y}{\dot{y}} \text{ (art. 38.) is } = \frac{y}{\dot{y}} \times \frac{by^2\dot{y}}{a^2 \times y^2 - a^2}^{\frac{1}{2}} =$$

$$\frac{by^3}{a^2 \times y^2 - a^2}^{\frac{1}{2}} = \frac{by^3}{a^2 v} = \text{the Subtangent CT.}$$

Construction. Through the center C draw the indefinite right line AT perpendicular to the ordinate CB; describe through the points A and B a semicircle ABH; make CI = the circumference of the circle ARA, and CK = aB; draw right lines HK and IL, and parallel to IL draw MB; lastly, produce CB to N, making CN = CM, and draw a right line NT parallel to KH: then will T be the point from which a Tangent to the point B

E c must

must be drawn. For, (AC) $a : y :: y : \frac{y^2}{a} =$
 CH, and (LC) $a : (CI) b :: (BC) y : CM =$
 $\frac{by}{a} = CN$; and (KC) $v : (CH) \frac{y^2}{a} :: (NC) \frac{by}{a}$
 $: CT = \frac{by^3}{a^2 v}.$

XI.

Fig. 91. *Quare* the Content of the cylindrical Ring
 ABDR?—the radius EA of the inner cir-
 cumference being given $=a$; and AD,
 the diameter of the ring, or of the gene-
 rating circle, $=b$.

Put any arch LA $=x$, and .78539 &c. $=c$:
 suppose Ed indefinitely near to ED; and de-
 scribe the concentric pricked circle BR, mak-
 ing AB $=BD = \frac{1}{2}b$. Then, the Moment of
 the ring, viz. Ad, will be equal to the area
 of the generating circle AD drawn into the
 Increment Bb. Now, EA : Aa :: EB : Bb,
 that is, $a : x' :: a + \frac{1}{2}b :: x' + \frac{bx'}{2a} = Bb$; and
 the area of the circle AD $=b^2c$: therefore,
 the Moment of the ring is $=b^2cx' + \frac{bx'^2}{2a} =$
 $b^2cx' + \frac{b^3cx'}{2a}$, or it's Fluxion (*art. 7.*) $=b^2c\dot{x}$
 $+$

$+\frac{b^3cx}{2a}$; the Fluent of which is $b^2cx+\frac{b^3cx}{2a}$

$=1+\frac{b}{2a}\times b^2cx=$ the Content of the Ring

from L to A; and therefore, by substituting $8ac$ for x , we have the Content of the whole

Ring $=1+\frac{b}{2a}\times 8ab^2c^2=b^2cx\overline{2a+b}\times 4c=$

the area of the generating circle AD drawn into the circumference of the pricked circle

BR. Q. E. I.

XII.

If a heavy Sphere, whose diameter is 4 inches, be let fall into a conical Glafs $\frac{1}{5}$ th full of Water, whose diameter is 5 inches and altitude 6; how much of the Sphere will be immersed in the Water?

Put the altitude $VH=6=a$; radius HF *Fig.*
 $=2.5=b$; $3.14159 \&c.$ (*art.* 122.) $=c$; 92.

AD, the diameter of the sphere, $=4=d$; and AC, that part of the said diameter under the water, $=x$; then, $CD=d-x$. Now, the capacity of the glafs is $=\frac{1}{3}ab^2c$; and therefore, by the question, the quantity of water in it is $=\frac{1}{3}ab^2c$. By 35 E. 3. $AC\times CD=CB^2$, that is, $dx-x^2=$ the square of

E e 2

the

the radius of the section of the sphere; therefore the area of the said section is $= cdx - cx^2$; which drawn into $\dot{x}$ is $cdx\dot{x} - cx^2\dot{x} =$ the Fluxion of the segment of the sphere under the water; the Fluent of which is $\frac{1}{2}cdx^2 - \frac{1}{3}cx^3 =$ the content of the said segment. Hence (similar solids being as the cubes of their homologous sides,) $\frac{1}{3}ab^2c : a^3 :: \frac{1}{15}ab^2c + \frac{1}{2}cdx^2 - \frac{1}{3}cx^3 : \frac{1}{5}a^3 + \frac{3a^2dx^2}{2b^2} - \frac{a^2x^3}{b^2} = VC^3$. But, $VF = \sqrt{a^2 + b^2}$; and $HF : FV :: GE : EV$, that is, $b : \sqrt{a^2 + b^2} :: \frac{1}{2}d : \frac{d}{2b}\sqrt{a^2 + b^2} = EV$; $\therefore VA (=VE - AE) = \frac{d}{2b}\sqrt{a^2 + b^2} - \frac{1}{2}d = 3.2$, which put $= e$; then $VC = e + x$, and $VC^3 = e^3 + 3e^2x + 3ex^2 + x^3 =$ (because by the above VC^3 is $=$) $\frac{1}{5}a^3 + \frac{3a^2dx^2}{2b^2} - \frac{a^2x^3}{b^2}$. Now, this equation produces $10a^2 + 10b^2.x^3 + 30b^2e - 15a^2d .x^2 + 30b^2e^2x = 2a^3b^2 - 10b^2e^3$, that is, $422.5x^3 - 1560x^2 + 1920x = 652$; which equation divided by 422.5 is $x^3 - 3.692x^2 + 4.544x = 1.543$; whence x may be found $= .546 =$

AC

AC= that Part of the Diameter of the Sphere under the Water. Q. E. I.

SCHOLIUM.

WE might now proceed to the investigation of the Centers of *gravity*, *percussion*, and *oscillation*, and a great variety of other Problems in the various branches of Mathematical and Philosophical Science: but, this Tract being intended as an *Introduction* only, for these Things we must refer the Learner to the larger and more extensive Books on the Subject*; in which, though he may meet with many Difficulties, it is hoped they are not such but he will now be able to surmount.

The Books, in *English*, professedly on the Subject, are,

1. A Treatise of Fluxions: or, an Introduction to Mathematical Philosophy. Containing a full explication of that Method by which the most celebrated Geometers of the pre-

* The Author particularly refers to the Works of his two celebrated Friends, Mr. *Emerson* and the late Mr. *Simpson*.

214 *An INTRODUCTION to the*

present Age have made such vast Advances in Mechanical Philosophy. By CHARLES HAYES, *Gent.*—Folio. 315 Pages. Cuts. 1704.

2. An Institution of Fluxions: Containing the first Principles, the Operations, with some of the Uses and Applications of that admirable Method. By HUMPHRY DITTON. —Octavo. 240 Pages. Cuts. 1706.

N. B. A Second Edition was printed in the Year 1726.

3. The Method of Fluxions, both Direct and Inverse. The former being a Translation from the celebrated Marquis *De L'Hospital's* *Analyse des Infiniments Petits*; and the latter supply'd by the Translator, E. STONE, F. R. S.—Octavo. 450 Pages. Plates. 1730.

4. The Doctrine of Fluxions, founded on Sir *Isaac Newton's* Method, published by himself in his Tracts upon the Quadrature of Curves. By JAMES HODGSON, F. R. S. and Master of the Royal Mathematical School in Christ's Hospital.—Quarto. 452 Pages. Cuts. 1736.

N. B. The Title-Page was reprinted in 1756; and, again, in 1758.

5. The

5. The Method of Fluxions and Infinite Series; with its Application to the Geometry of Curve-Lines. By the Inventor Sir *Isaac Newton*, K^t. late President of the Royal Society. Translated from the Author's Latin Original, not yet made public. To which is subjoined, a Perpetual Comment upon the whole Work. By JOHN COLSON, M.A. and F.R. S.—Quarto. 339 Pages. Cuts. 1736.

N. B. The same Piece was translated by another Hand; and published, without a Comment, in the Year 1737. Octavo. 189 Pages. Cuts.

* * * The Original was written in the Year 1671; but founded on a smaller manuscript Tract composed in *November* 1666; in which the great Inventor used the same Method of noting the Fluxions of variable Quantities as that which he afterwards generally followed, that is, *Pointing*.

6. A Mathematical Treatise: Containing a System of Conic-Sections; with the Doctrine of Fluxions and Fluents, applied to various Subjects; viz. To the finding the Maximums and Minimums of Quantities; Radii of Evolution, Refraction, Reflection; superficial and solid Contents of curvilinear Figures; Rectification of Curve-lines; Centers

ters of Gravity, Oscillation and Percussion: as also, to the Resolution of a select Collection of the most useful, and many new, Physico-Mathematical Problems. By JOHN MULLER.—Quarto. 227 Pages. Plates. 1736.

7. The Doctrine and Application of Fluxions. Containing (besides what is common on the Subject) a number of *new* Improvements in the Theory; and the Solution of a variety of *new* and very interesting Problems in different Branches of the Mathematics. By THOMAS SIMPSON, F. R. S.—2 Volumes, Octavo. 576 Pages. Cuts. 1750.

* * * This Work is, perhaps, not inferior to any on the Subject.

N. B. He published a *Treatise of Fluxions* in the Year 1737. Quarto. 216 Pages. Cuts.

††† This great and penetrating Genius was born *August* the 20th, 1710, and died *May* the 14th, 1761.

8. A Treatise of Fluxions. By COLIN MAC LAURIN, A. M. Professor of Mathematics in the University of Edinburgh, and F. R. S.—2 Volumes, Quarto. 754 Pages. Plates. 1742.

* * * In

DOCTRINE of FLUXIONS. 217

* * In this masterly Work, the Subject is handled agreeable to the Method of reasoning used by the ancient Mathematicians.

††† This celebrated Writer was born in *February* 1698, and died *June* the 14th, 1746.

9. The Doctrine of Fluxions: not only explaining the Elements thereof, but also its Application and Use in the several Parts of Mathematics and Natural Philosophy. By W. EMERSON. The Second Edition, corrected and greatly enlarged.——Octavo. 432 Pages. Plates. 1757.

N. B. The First Edition of this elegant and excellent Work was printed in the Year 1743. Octavo. 300 Pages. Plates.

10. Sir *Isaac Newton's* two Treatises, of the Quadrature of Curves, and Analysis by Equations of an infinite number of Terms, explained. Containing the Treatises themselves, translated into English, with a large Commentary. By JOHN STEWART, A. M. Professor of Mathematics in the Marishal College and University of Aberdeen.——Quarto. 479 Pages. Cuts. 1745.

††† The Analysis by Equations was first written before the Year 1669; and the Quadrature
F f of

218 *An* INTRODUCTION, &c.

of Curves before the Year 1676: But, the last Scholium in the Quadratures was added but just before the Tract was published, which was by the great Author himself in the Year 1704. The Analysis was first printed in 1711.

11. The Method of Fluxions applied to a select number of useful Problems. By NICHOLAS SAUNDERSON, L.L.D. Late Professor of Mathematics in the University of Cambridge.—Octavo. 309 Pages. Plates. 1756.

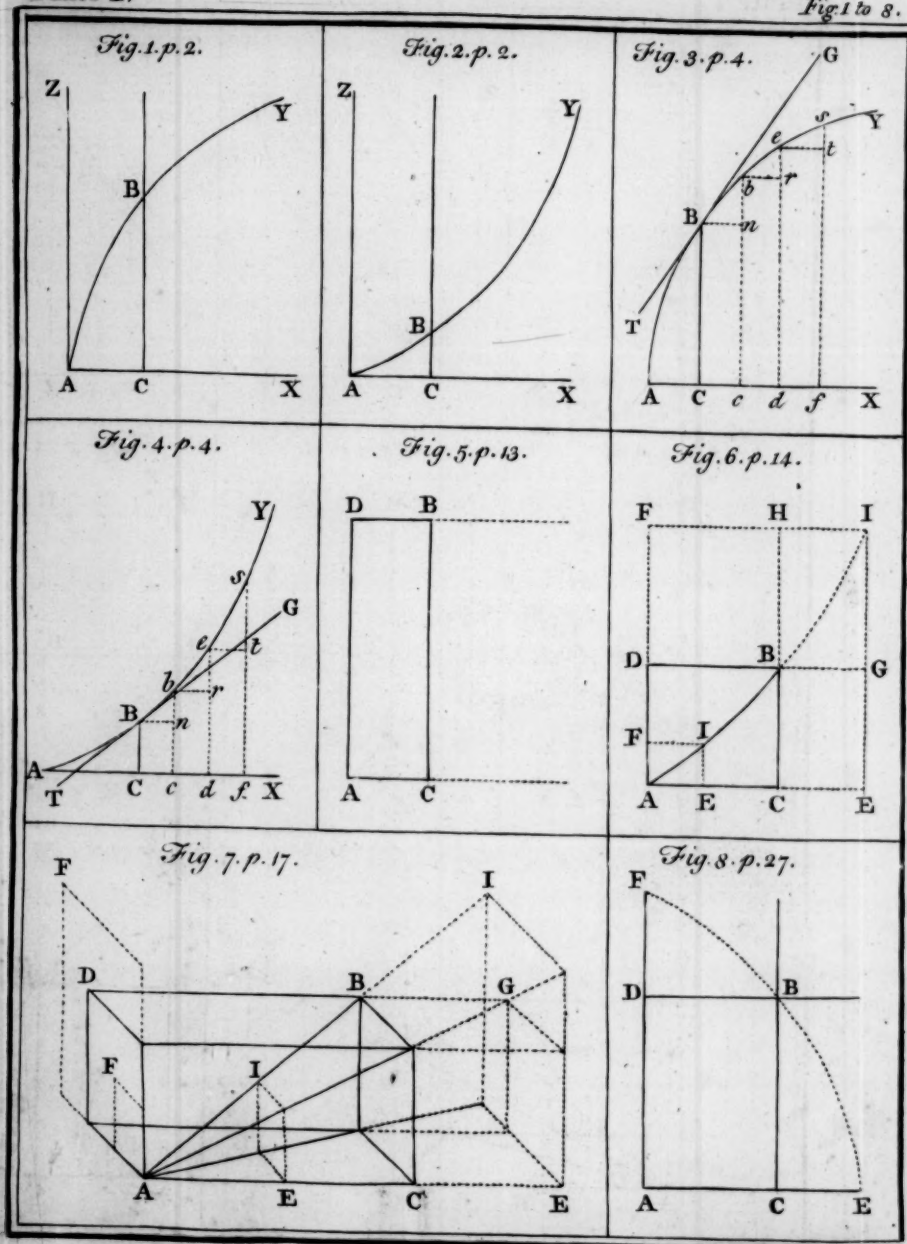
12. A Treatise of Fluxions. By ISRAEL LYONS, Junior.—Octavo. 269 Pages. Plates. 1758.

F I N I S.

18 AP 68

Plate I.

Fig. 1 to 8.



J. M. G. sc.

Plate II.

Fig. 9 to 17.

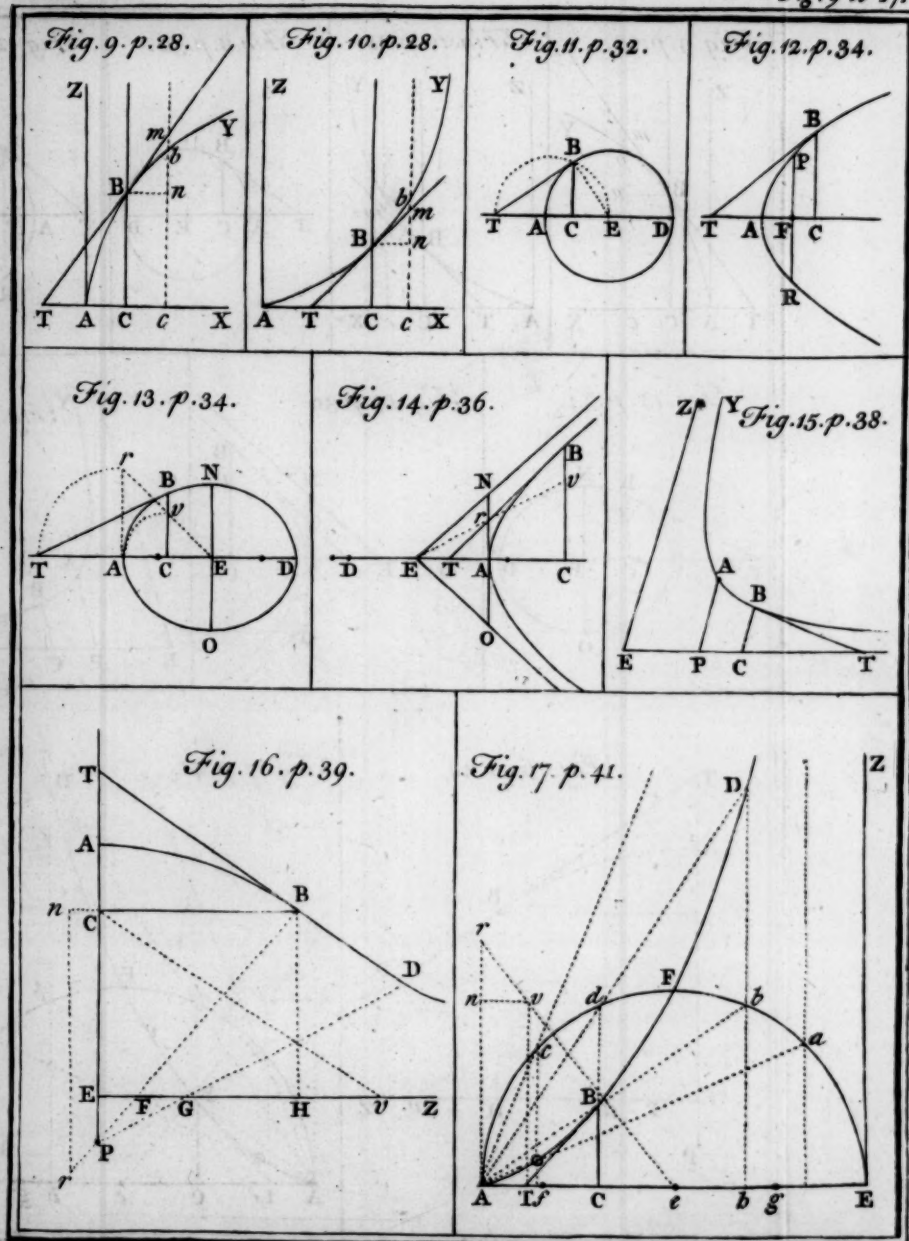


Fig. 18. p. 43.

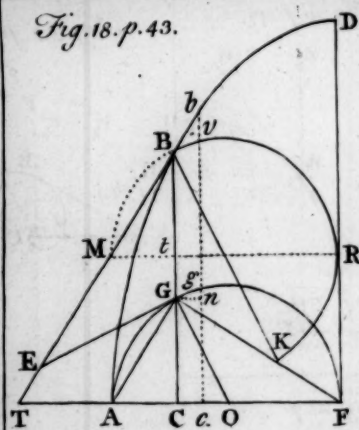


Fig. 20. p. 47.

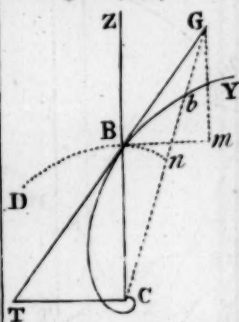


Fig. 22. p. 51.

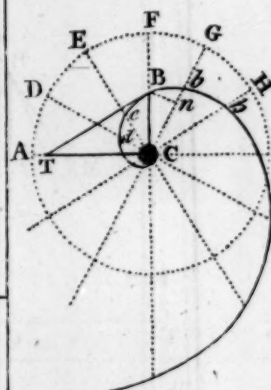


Fig. 21. p. 50.

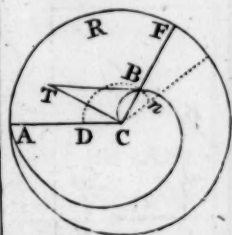


Fig. 19. p. 46.

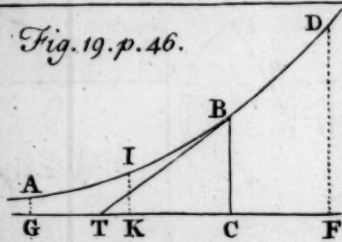


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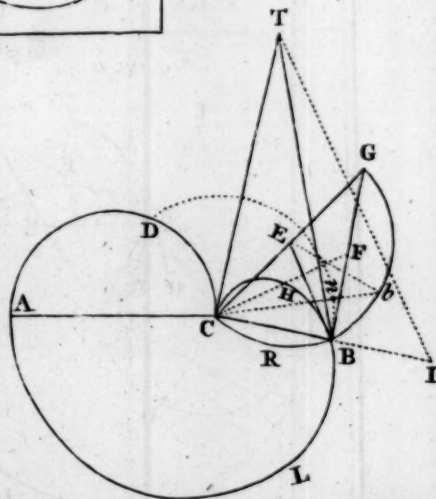
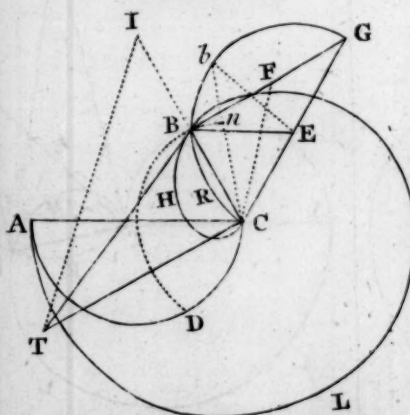


Fig. 23. p. 54.



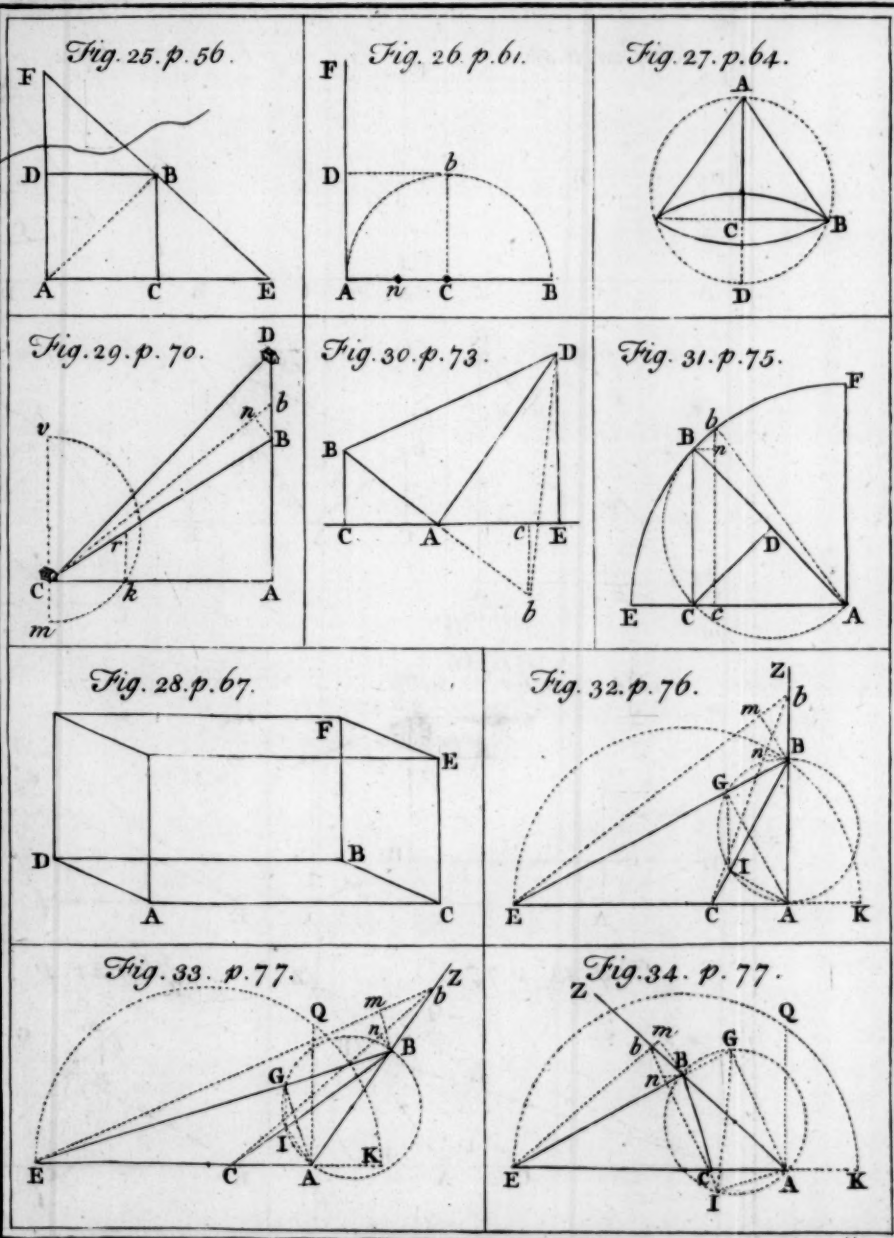


Fig. 35. p. 79.

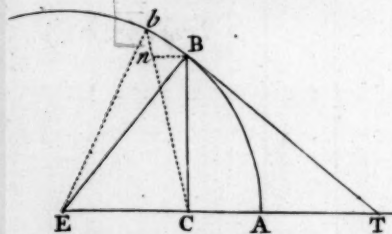


Fig. 37. p. 83.

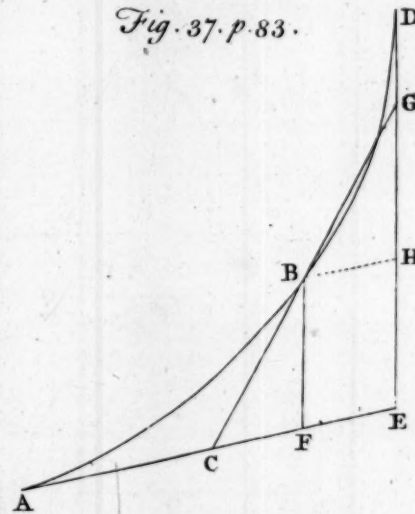


Fig. 36. p. 81.

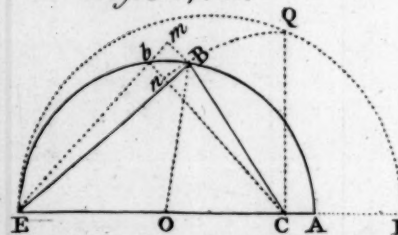
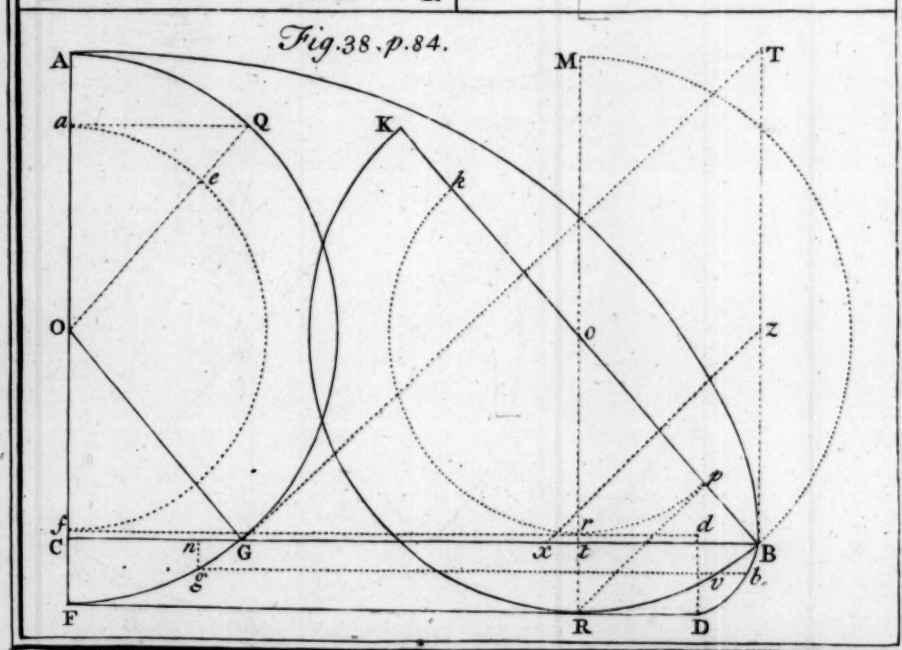


Fig. 38. p. 84.



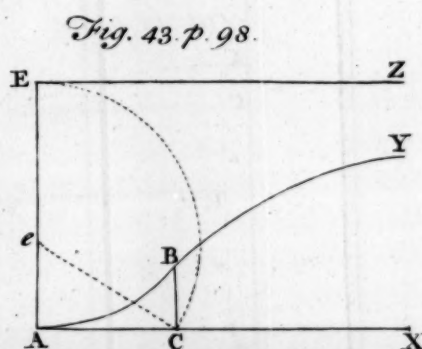
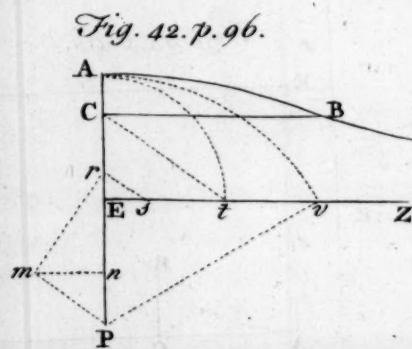
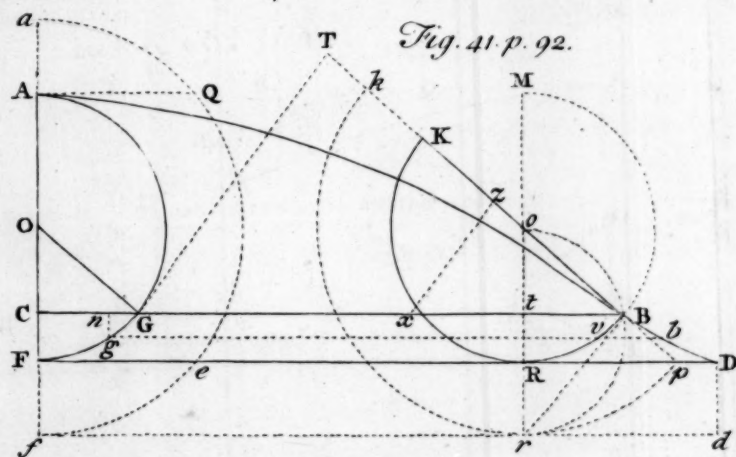
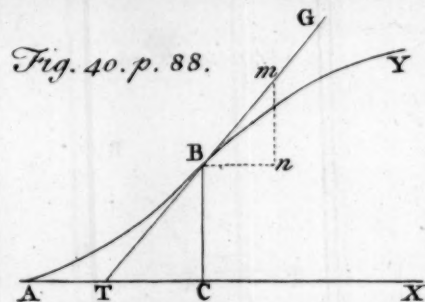
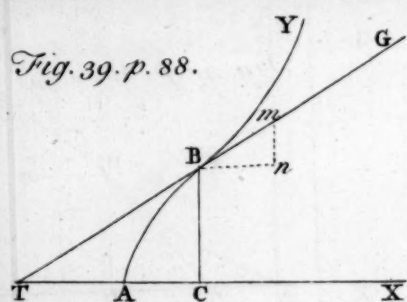


Fig. 44. p. 101.

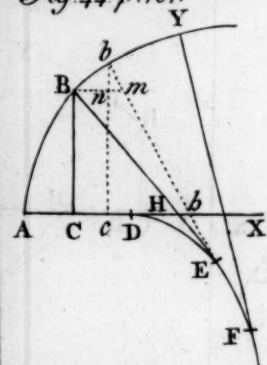


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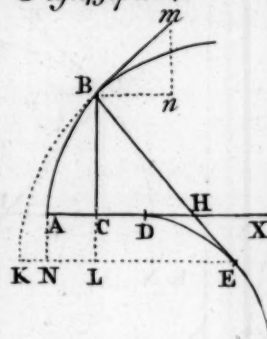


Fig. 46. p. 108.

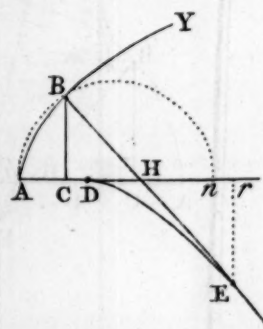


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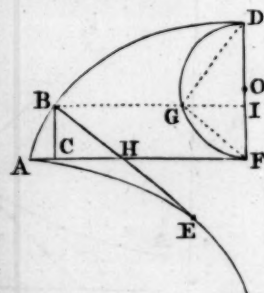


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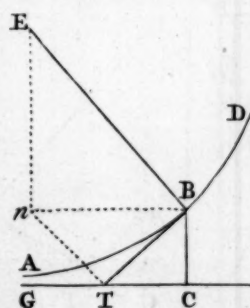


Fig. 49. p. 115.

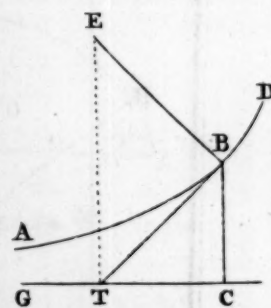


Fig. 50. p. 117.

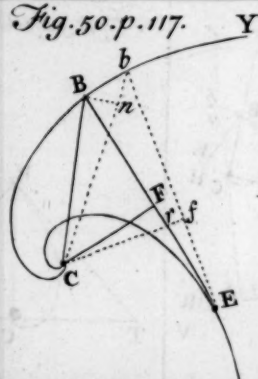


Fig. 51. p. 119.

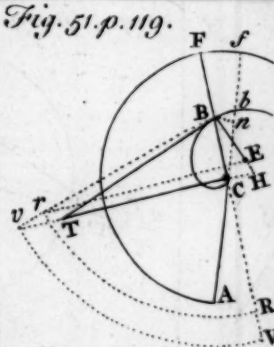


Fig. 52. p. 122.

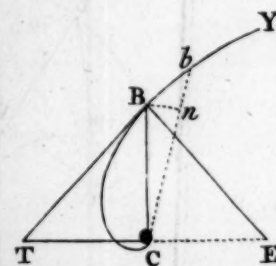


Fig. 53. p. 124.

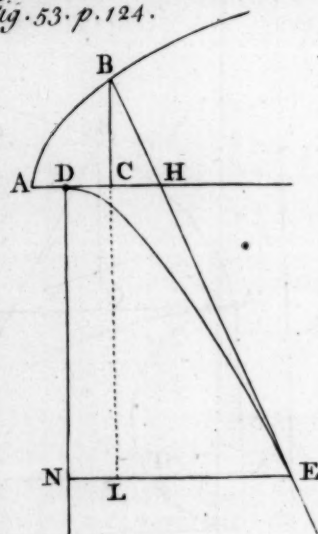


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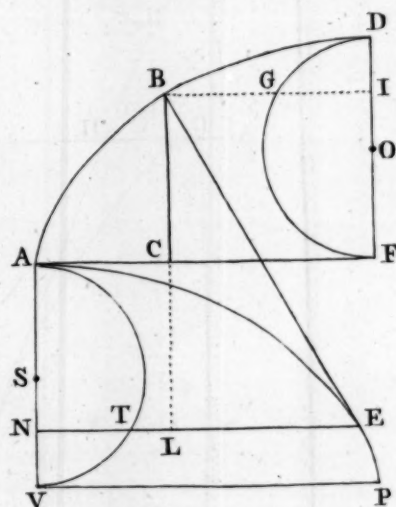


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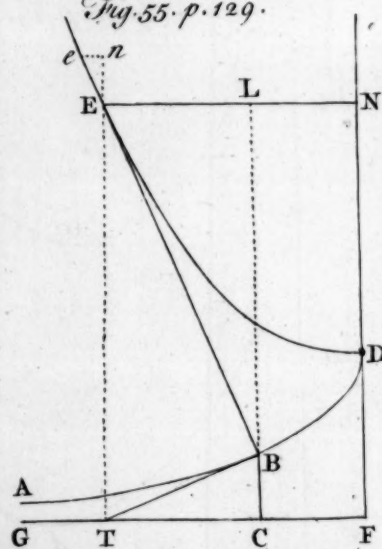
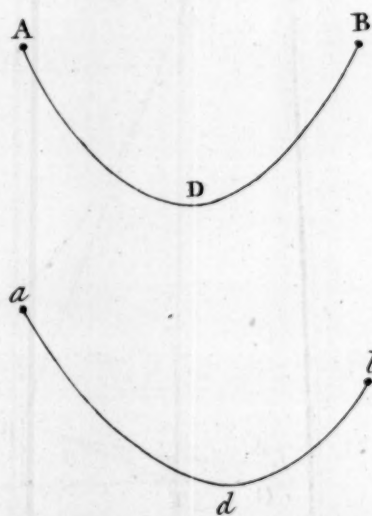


Fig. 56. p. 131.



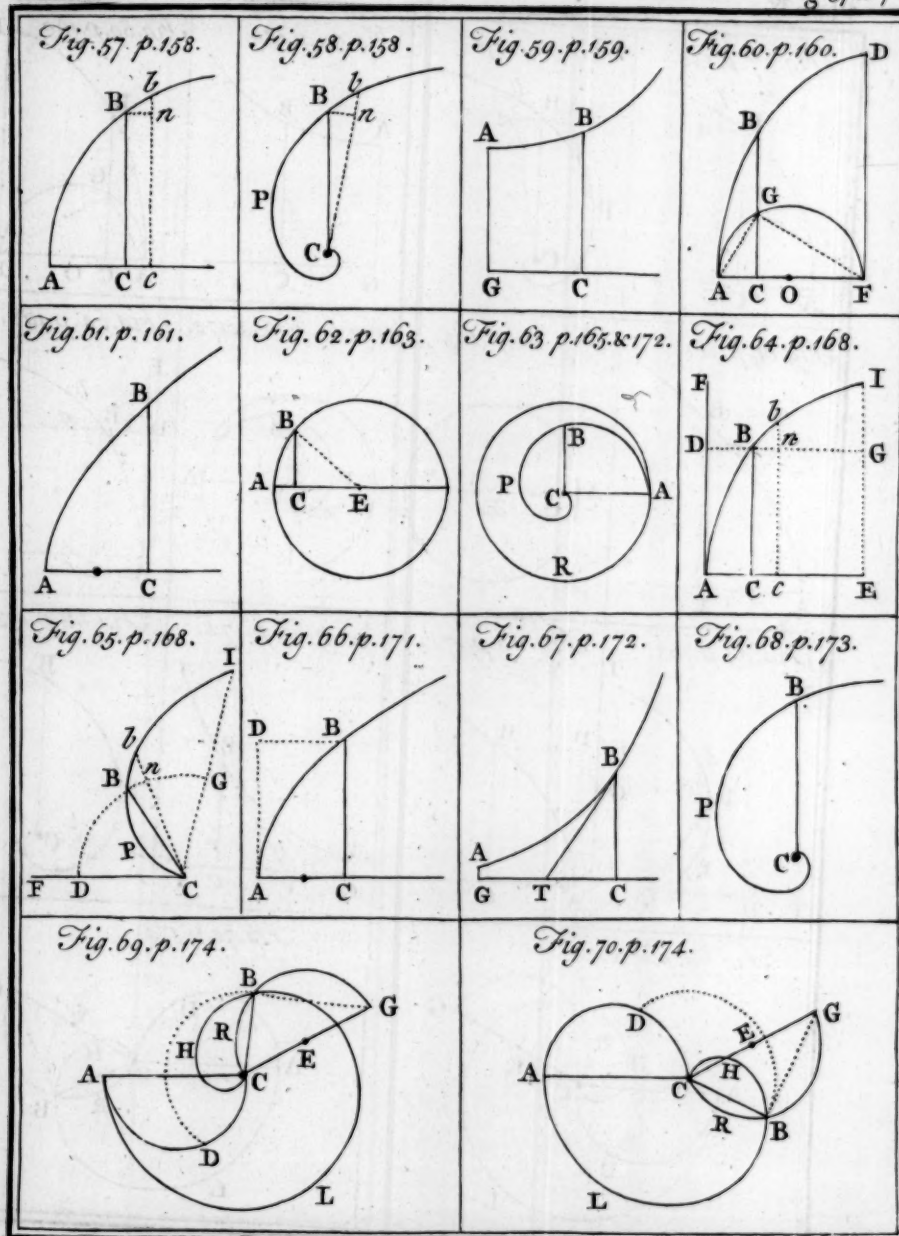


Fig. 71. p. 175.

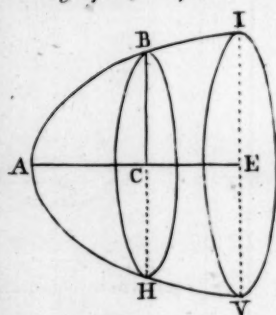


Fig. 72. p. 177.

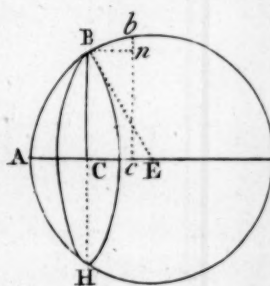


Fig. 73. p. 178.

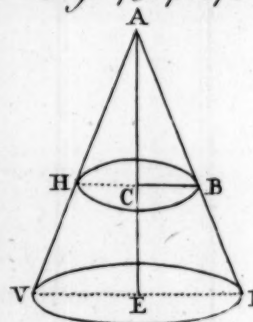


Fig. 74. p. 179. & 184.

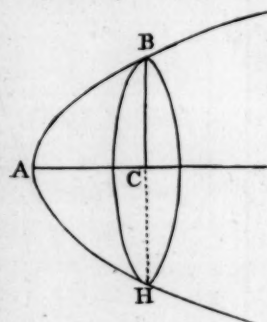


Fig. 75. p. 180.

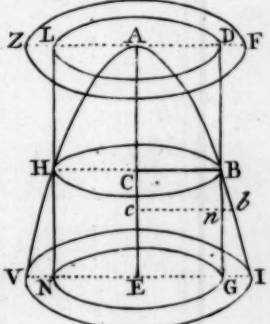


Fig. 78. p. 187.

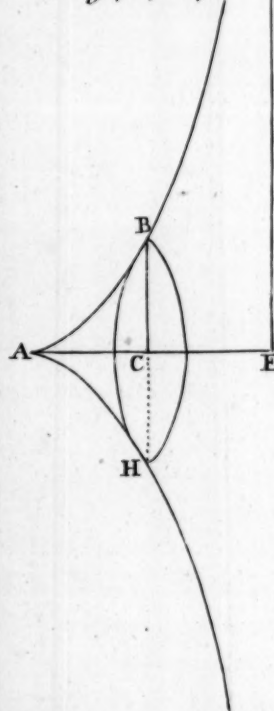
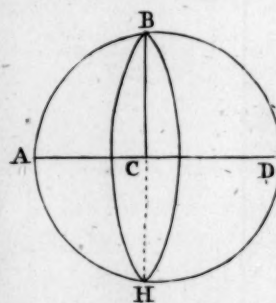


Fig. 76. p. 183.



A Fig. 77. p. 185.

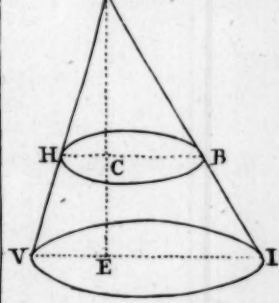


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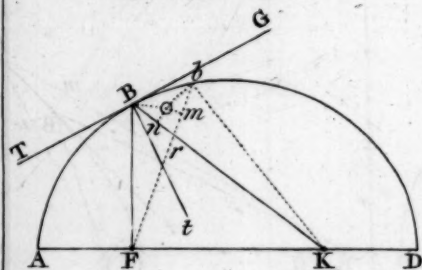


Fig. 80. p. 190.

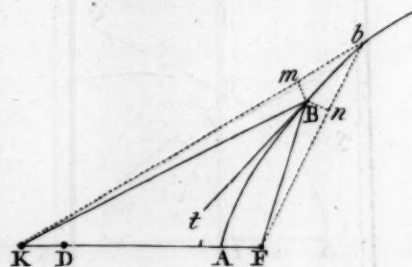


Fig. 81. p. 191.

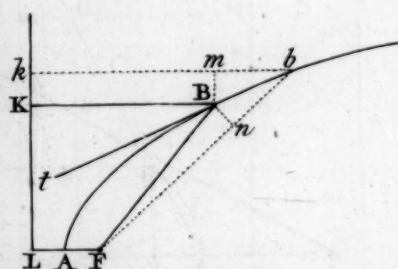


Fig. 82. p. 192.

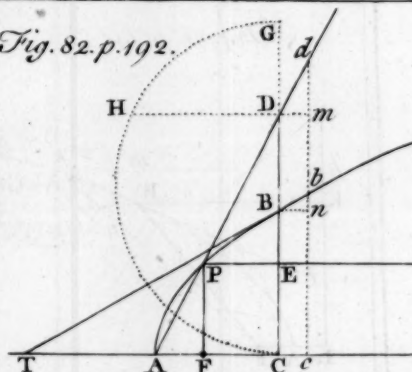


Fig. 83. p. 193.

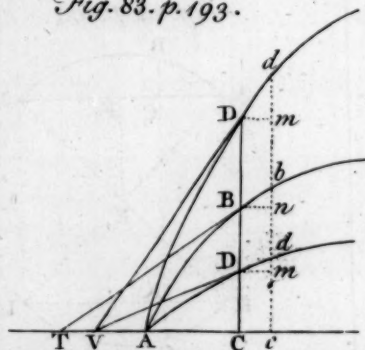


Fig. 84. p. 194.

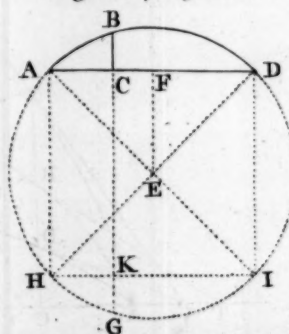


Fig. 85. p. 197.

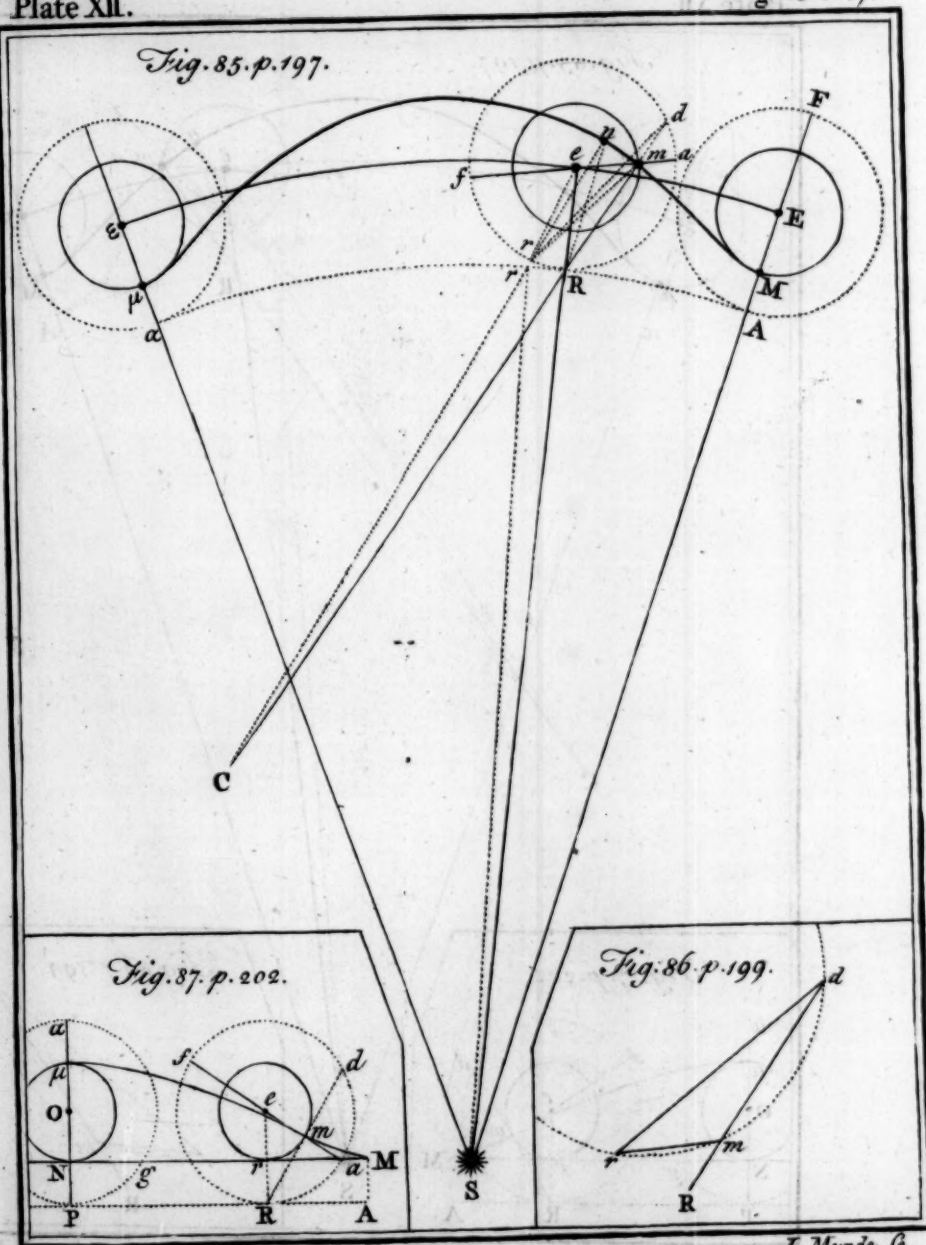


Fig. 87. p. 202.

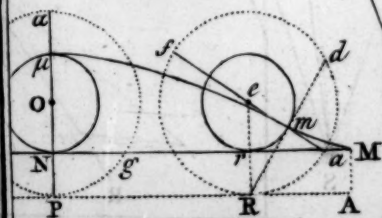
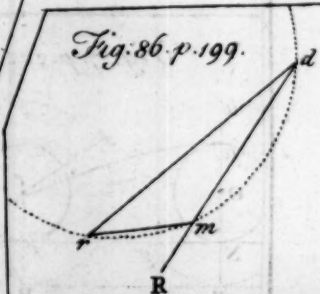
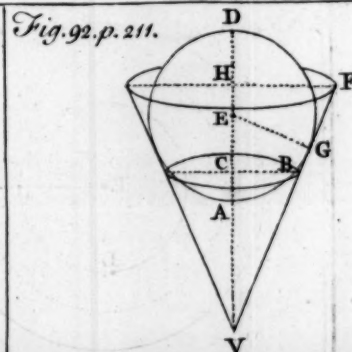
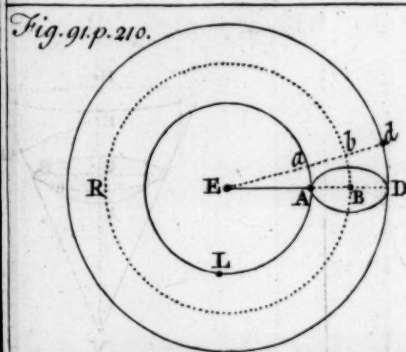
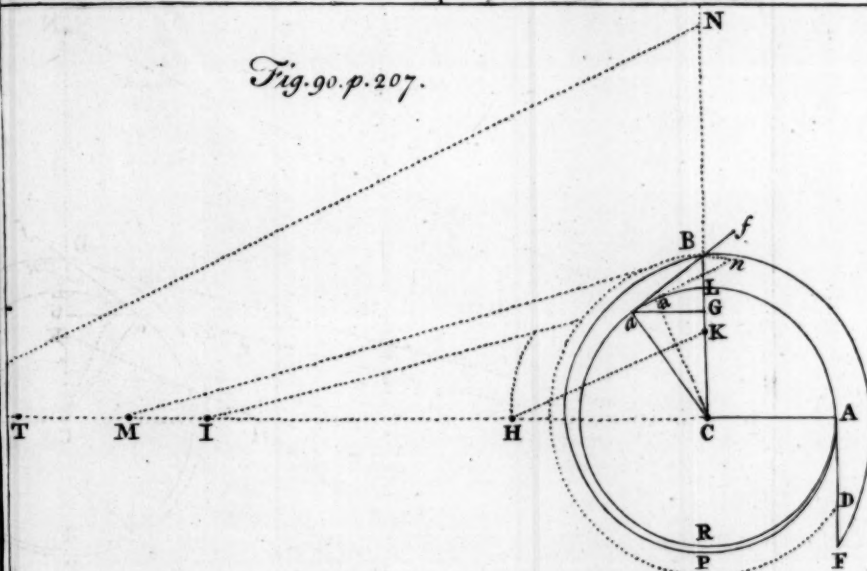
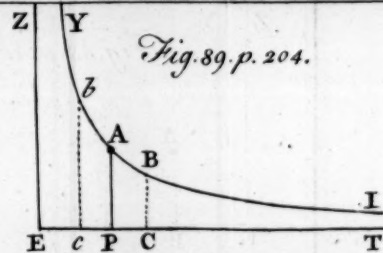
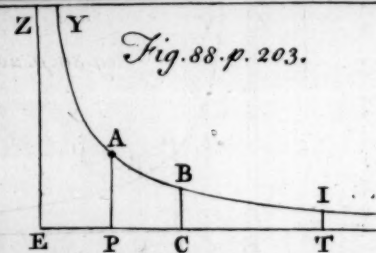


Fig. 86. p. 199.







Notes, &c.

Page 58. "any Negative Power of a Maximum will be a Minimum; and any Negative Power of a Minimum will be a Maximum". —

This Remark has not been made by any other Writer, I believe.

Page 16. 2°. This New Way of demonstrating the Truth of the Rule for finding the Fluxion of xyz has been copied by my Friend Mr. Benjamin Martin in his Mathematical Institutions.

JR.